Proceedings of the seventh EWM meeting

EUROPEAN WOMEN IN MATHEMATICS
Universidad Complutense de Madrid
September 4–9, 1995

Contents

Preface to the printed edition

This volume contains records of the mathematical activities and some of the other activities which took place during the seventh general meeting of European Women in Mathematics at Universidad Complutense de Madrid, September 4 – 9, 1995.

The meeting was attended by 46 participants from 14 countries (Denmark, England,Finland, France, Germany, Italy, Norway, Portugal, Rumania, Russia, Spain, Sweden and Switzerland).

The main organizers were by far Capi Corrales Rodrigáñez and Raquel Mallavibarrena from Madrid. They were assisted by an organizing committee consisting of Bodil Branner (Denmark), Isabel Labouriau (Portugal), Rosa Maria Miró–Roig (Spain), Marjatta Näätänen (Finland), Sylvie Paycha (France), Caroline Series (England), Laura Tedeschini–Lalli (Italy).

In organizing the meeting we build on earlier experiences. In particular the work done by Eva Bayer, Michèle Audin and Catherine Goldstein (all from France) around the fifth EWM meeting in Luminy in December 1991 has been a constant source of inspiration.

We are very grateful for the financial support we received from the spanish Instituto de la Mujer and the Ministerio de Educación y Ciencia, both for the Madrid meeting and for the publication of these Proceedings.

The logo of the EWM meeting and the t-shirt pattern (shown on the front page) was designed by DODOT.

These Proceedings were edited by Bodil Branner and Núria Fagella. We thank Christian Mannes for setting up the TeX style we have used.

The photos were taken by Marketa Novak.

We wish to express our thanks to all participants for making the Madrid meeting a success and to all who contributed in writing to this volume. Especially we thank Capi and Raquel for making it all possible.

February, 1996
Bodil Branner and Núria Fagella.

Addendum to the electronic edition

The original version of these proceedings was printed at the Universitat de Barcelona, with *Dep. Legal L 544 - 1996, Impreso Poblagrafic S.L. Av. Estacion s/n Pobla de Segur.*

The main difference between the electronic edition and the printed one is that some misprints have been corrected and that the logo designed by DODOT, the photos and the figures contained in the mathematical papers do not appear here.

If you wish to get a copy of the printed version, please contact your regional coordinator and ask if some are still available.

European Women in Mathematics

EWM is an affiliation of women bound by a common interest in the position of women in mathematics. Our purposes are:

– To encourage women to take up and continue their studies in mathematics.

– To support women with or desiring careers in research in mathematics or mathematics related fields.

– To provide a meeting place for these women.

– To foster international scientific communication among women and men in the mathematical community.

– To cooperate with groups and organizations, in Europe and elsewhere, with similar goals.

Our organization was conceived at the International Congress of Mathematicians in Berkeley, August 1986, as a result of a panel discussion organized by the Association for Women in Mathematics, in which several European women mathematicians took part. There have since been six European meetings: in Paris (1986), in Copenhagen (1987), in Warwick (England) (1988), in Lisbon (1990), in Marseilles (1991), in Warsaw (1993), and in Madrid (1995). The next meeting will be in 1997. The place of the meeting will be announced later.

At the time of writing, there are participating members in the following countries:

Belgium, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Italy, Latvia, Lithuania, the Netherlands, Norway, Poland, Portugal, Romania, Russia, Spain, Sweden, Switzerland, Turkey, Ukraine, and the United Kingdom; contacts in Albania as well as with several non-European countries. Activities and publicity within each country are organized by regional co-ordinators. Each country or region is free to form its own regional or national organization, taking whatever organizational or legal form is appropriate to the local circumstances. Such an organization, Femmes et Mathematiques, already exists in France. Other members are encouraged to consider the possibility of forming such local, regional or national groups themselves. There is also an e-mail network.

For further information contact:

The secretary of EWM: Riitta Ulmanen,
Department of Mathematics,
P.O.Box 4 (Yliopistonkatu 5),
FIN - 00014, University of Helsinki, Finland;
e-mail: ulmanen@sophie.helsinki.fi,
Tel 358 9 191 22853, Fax 358 9 191 23213

For details of the e-mail network contact:
sarah.rees@newcastle.ac.uk

September, 1995

EWM on the Web

EWM now has a page on the Web. There are two adresses, although both contain the same information. One in Helsinki

http://www.math.helsinki.fi/EWM

and one in Austria

http://www.risc.uni-linz.ac.at/misc-info/ewm/EWM.html

The Web account has been set up by Olga Caprotti, Giovanna Roda, Ileana Tomuta and Daniela Vasaru at

RISC – Linz
Research Institute for Symbolic Computation
Johannes Kepler University
A – 4040 Linz, Austria

They can be reached at the EWM – Web account

ewm@risc.uni-linz.ac.at

or at their personal accounts

FirstName.LastName@risc.uni-linz.ac.at

History Since Luminy

EWM since its fifth meeting in Luminy

Caroline Series
University of Warwick, England

The fifth EWM meeting was held in Luminy, France, December 9-13th, 1991. In the lengthy report which was published of the Luminy meeting, I wrote a brief history of the first five years of EWM. Since this is the first major EWM report to be published since that time, I have been asked to write a few words to record what has happened in the interim.

1. The Sixth General Meeting, Warsaw, June 7th-11th 1993

The sixth General EWM Meeting took place at the Technical University in Warsaw from June 7th-11th 1993. The main organiser was Anna Romanowska. The meeting was attended by about 60 participants from 16 European countries. The conference featured lectures in pure and applied mathematics, an interesting session on creativity, and several business sessions. There was a report from the Round Table on women in Mathematics at the European Congress in 1992. We learned that the proportion of women among doctorates in mathematics is highest in eastern European and Mediterranean countries (Greece, Italy, and Spain were represented at the conference), and lowest in Scandinavia, the Netherlands, the UK, and Germany. There has been much speculation about this somewhat counter-intuitive situation, but it seems more productive to concentrate on how to get more women into mathematics in the northern countries. Mary Gray gave a talk on the history, aims and activities of the AWM, and also gave helpful advice about the draft statutes of EWM.

2. Legalisation of EWM and establishment of the Helsinki Office, 1991-1994

One of our early failures as an organisation, which I mentioned in the report of the Luminy meeting, was our loose structure in which everything depended on just one person, the international coordinator. By the time of the Luminy meeting we had decided that it was time to set in place some more formal organisation, and that we wanted EWM to be a legal body. During the Luminy meeting we did a lot of work drafting our basic organisational structure and statutes and a small committee, consisting of Marjatta Näätänen, Riitta Ulmanen and myself, was formed to carry this work further.

In the course of discussing our statutes we were forced to consider the structure and function of EWM in great detail. We wanted to make EWM work by consensus, but at the same time we had learned from experience that it is vital to have a core of central people responsible for the continuity and smooth functioning of the organisation. This we achieved by setting up a standing committee, led by a convenor, to deal with executive matters and in particular in planning the next meeting; regional coordinators to deal with members and circulate information in their regions; and international coordinators to watch over and circulate information among the regions. We also established membership procedures and a method of collecting dues. None of this is easy in an international organisation with members living in many different circumstances.

At the Warsaw meeting, the General Assembly accepted, with some changes, the statute prepared by our committee in consultation with Finnish lawyers. It was here that the fine details of the structure of membership and fees were hammered out. It was decided to formalise membership and start collecting dues from 1994 to pay for the work in Helsinki and our international activities. There are three rates to allow for the many different circumstances in which we live. The regional co-ordinators are responsible for collecting this money in local currency and sending it to the general EWM bank account which is held in Helsinki.

The establishment of EWM as a legal body was finalised on December 2nd 1993. This was surely an important marker in the history of EWM.

One reason for the choice of Helsinki as the legal seat of EWM was that Helsinki was already the seat of the European Mathematical Society (EMS). We were very fortunate that Riitta Ulmanen agreed to be our general secretary. Riitta is librarian in the Mathematics Department in the University of Helsinki. We made an application form for EWM membership which regional co-ordinators circulate and collect yearly. The office in Helsinki is an information centre and it collects and keeps constantly updated information about members, finances, committees and coordinators. Riitta also answers enquiries about EWM and mails information to members, usually via the regional coordinators . For a scattered organisation like ours it is crucial to have a central place where everything is kept together. For the last two years, Marjatta has obtained funding from the Finnish Ministry of Education and the Finish Cultural Fundation to support Riitta's work.

The address and telphone numbers of the EWM Helsinki office have changed recently and the new address is :

EWM Office, Riitta Ulmanen , Secretary
Department of Mathematics, PO Box 4
Yliopistonkatu 5, FIN-00014
University of Helsinki, Finland
Tel 358 0 191 22853 – Fax 358 0 191 23213
e-mail: ulmanen@sophie.helsinki.fi

3. The European Mathematical Society Committee on Women and Mathematics

The European Mathematical Society (EMS) was founded in October 1990. Eva Bayer was instrumental in setting up and chairing the EMS Committee on Women and Mathematics from January 1991. Not surprisingly, EWM members, particularly Eva, have played a prominent rôle. The committee organised a round table at the first European Congress of Mathematics in Paris, 1992, which had about 150 participants, 5 short talks, and a lively and interesting discussion.

The committee made an analysis of the situation of women mathematicians in Germany, which has one of the lowest proportions of women to men among mathematicians in Europe. The results, together with a report based on the discussions of the round table, appear in the 1992 Proceedings of the ECM.

In spring 1994, the EMS committee made an extensive inquiry in Switzerland about the low number of women mathematicians in that country. The results were discussed at the International Congress of Mathematicians in Zürich, August 1994. EWM and the EMS committee also organised a more general discussion about the countries in European countries with an unusually low proportion of women mathematicians.

At the Zurich ICM, there was a very nice "Noether lecture" by the distinguished Russian woman mathematician O.A. Ladizhenskaya. There was also a panel discussion jointly organised by AWM, the Canadian women mathematician's group, and EWM. Reports of these events may be found in our January 1995 Newsletter Number 2.

The next EMS meeting is to be held in Budapest in summer 1996. Kari Hag is organising a round table on the topic "Females in Mathematics in the Iberian and Scandanavian Peninsulars". A pleasing number of women speakers have been invited, including Dusa McDuff who is to give a plenary session.

4. Regional Meetings

We now have regional coordinators in Belgium, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Italy, Latvia, Lithuania, Netherlands, Norway, Poland, Portugal, Roumania, Russia, Spain, Sweden, Switzerland, Turkey, Ukraine, United Kingdom.

There have been regional meetings of women mathematicians in, among others, France, Germany, Russia, Sweden and the UK. *Femmes et mathématiques* continues to be an important and very active organisation in France. During 1995, *femmes et mathématiques* organised three general assemblies, two in Paris and one in Lille. On 8th March 1995 -International women's day- there were actions in seven universities on the theme *Women in mathematics*. In 1996, several events are planned including a one day forum for young women mathematicians in Paris in January and a general assembly in March in Rennes. The Russian Women Mathematicians Association (RMWA) was founded at a conference in May 1993 in Suzdal and already has more than 300 members from more than 40 cities of Russia and the FSU. A second International conference took place in Voronezh in May 1995 and the third is planned for Volgograd in May 1996 (see below). An EWM group, AIDIM, associazione italiana donne in matematica, has been formed in Italy which has about 30 official members with more interested. This group took part in a congress in Anacapri in 1994 where it presented its purposes and proposed some topics for discussion. The British group BWM organised a very successful one day meeting in London in September 1995 attended by 50 women from all over the UK. There is also a functioning inter-uk email network.

The German group is also active: there is an e-mail net managed from Magdeburg with about 150 participants, where there is also an ftp-server with information about EWM. A data-base of women mathematicians with a Habilitation in Germany since Emmy Noether has been set up. There have been some local meetings; in 1994 there was a "Women and Mathematics" meeting in Oberwolfach organized by Catherine Bantle (Basel) in which a large EWM group participated.

5. The e-mail network

For some time we have relied heavily for our communications on the email network set up by Laura Tedeschini–Lalli in Italy. This is a very easy and efficient method of communicating and saves much time and expense. In Madrid, Sarah Rees offered to reorganise and administer the network from her university in Newcastle. She has now set up a new network and hopes eventually to have subnets for each individual country or region. To join, mail her at sarah.rees@newcastle.ac.uk

6. The Newsletter

We have been talking for a long time about starting a newsletter, and three issues, roughly one per year, have now appeared. The newsletter is being edited by Cathy Hobbs and Marlen Fritsche. It is distributed via the regional coordinators and in addition is sent by email, in Tex and plain versions , to all on our network. You are sent a copy automatically on joining the network, otherwise copies hard or electronic can be obtained from the Helsinki office.

7. The Seventh General Meeting, Madrid, September 4-9th, 1995

Since the rest of this report is about the Madrid meeting, I shall not say anything here. I would just like to note that we were very pleased that Capi Corrales managed to get funding for a planning meeting of the organising committee, including five non-Spanish members, who came to Madrid in November 1994. Hosted by Capi in her spacious apartment, the committee was able to spend an entire whole weekend, from early till late, discussing and planning all aspects of the conference. This meant that the wider international committee could take a really active and informed part in organising the meeting and were much better able to support the Spanish organisers throughout the process. This kind of support for the local organisers is invaluable and ideally we should try to have such a pre-meeting before every large meeting.

8. The future

One of our main aims should certainly be to continue to develop more regional activities. These are easier and cheaper to organise than big international events, and can be attended relatively easily by members who cannot get to the international meetings. They can relate to local needs and are in people's own language. Already a number of activities of this kind are lined up for the coming year. Besides the Budapest meeting mentioned above, EWM is organising, jointly with *femmes et mathématiques*, an interdisciplinary two day workshop on Renormalisation from June 14th-15th 1996 in Paris. There will also be a joint Franco-Russian meeting organised by femmes et mathématiques and the Russian Association for Women Mathematicians (RAWM) in Marseille in December 1996. The British group BWM is planning another one day meeting in London next September. The third major meeting of the Russian Association is planned for May 27-31, 1996 in Volgograd. It will part of an international forum on "Problems of survival" under the title "Mathematics, modelling and ecology". For details contact the organiser Prof. G. Riznichenko, riznich@orgmath.msk.su. The German group is planning a long weekend in June 1996.

On the international level, to facilitate the flow of information, we are hoping to set up an Internet page.

One of our main problems is money, and if we could find some funding on a steady basis it would take a huge burden off the organisers. Funding meetings is a big problem which ideally should be taken care of well in advance. We feel we should be able to get more funding from the EC, but it takes hard work and persistence to put in applications. We badly need people to help in this work. We also need more people to sign up as members and pay their dues so that we can be more active and have more secure ongoing support for our international work.

As EWM becomes more established, it is very pleasing to see that many people already see us as an established smoothly running organisation. They expect that EWM will be able to answer queries, produce statistics, organise conferences, and put in place networks and support systems and opportunities for women to meet. Of course we are very glad to be seen in this rôle, and this is indeed exactly what our original vision was about, but keeping it all going is still hard work. We badly need more people to come forward to take part in running the organisation. There are many jobs, small and large, to be done. Although hard work, this has many rewards as one gets to know and cooperate with women mathematicians from all over Europe, and perhaps it is really the best way to find out what our network is all about.

In short, although there are still many problems and shortcomings in our organisation, we are spreading, I believe that people are beginning to take us for granted, and that means that we have arrived to stay!

Brief report on the sixth meeting of EWM in Warsaw

June 7th-11th, 1993

The meeting was attended by about 60 participants from 16 European countries: Denmark, Finland, France, Germany, Greece, Italy, Lithuania, the Netherlands, Norway, Poland, Portugal, Russia, Spain, Sweden, Turkey, Ukraine. Mary Gray from the USA came as a representative of the Association for Women in Mathematics, of which she is one of the founding members. The main organiser of the meeting was Anna Romanowska of the Technical University, Warsaw.

1. Mathematical Programme

There were mathematical talks by

- Danuta Preworska-Rolewicz (Warsaw): "Differential calculus as calculus";
- Janina Kotus (Warsaw): "Fractals arising in holomorphic dynamical systems";
- Viviane Baladi (Lyon): "Some recent results on random perturbations of dynamical systems";
- Krystyna Kuperberg (Auburn, Florida): "The recent revival of continuum theory";
- Ina Kersten (Bielefeld): "Linear algebraic groups";
- Zofia Adamowicz (Warsaw): "On real closed fields".

There was also a poster session in which participants presented aspects of their work.

2. General Discussions

As decided at the previous EWM meeting, the non-mathematical theme was Creativity, organised by Coby Geijsel. There were two talks:

- Coby Geijsel (Amsterdam): "Images of creativity"
- Karin Kwast (Amsterdam): "Creativity, a case study"

These were followed by discussions in small groups. Other activities were a report on the Round Table on "Women in Mathematics" which took place at the European Mathematical Congress in Paris 1992; a talk by Mary Gray on the history, aims and activities of the AWM; and one by Krystyna Kuperberg, a Polish mathematician who has settled in the USA, comparing the academic environment for women mathematicians in Poland, Sweden and the USA. There was also a discussion on the situation of women mathematicians in the former socialistic countries. Moreover the programme included a talk by Vassiliki Farmaki (Athens): "Women mathematicians in Ancient Greece", and a talk by Magdalena Jaroszewska (Poznan): "Olga Taudsky-Todd".

3. Organisation of EWM and the General Assembly

Following the decision in Luminy to go forward with the establishment of EWM as a legal body, much work had been done, mainly by Caroline Series and by Marjatta Näätänen and Riitta Ulmanen of Helsinki in consultation with Finnish lawyers, on preparing a draft of the statutes. This draft was presented to the general assembly and, following detailed discussion, the essentials were accepted with some changes. There will be two categories of membership, supporting members and full members. Supporting members can come to meetings but not vote, and men can only join as supporting members. The suggestion of this formulation was made by Mary Gray, who in addition to her long association with AWM is a lawyer as well as a mathematician. The legalisation committee was asked to prepare a final version of the statutes and the legalisation was completed by December 2nd of 1993. The legal seat of EWM will be in Helsinki in Finland where Riitta Ulmanen as the secretary will have an office for EWM at the Department of Mathematics at University of Helsinki. There was considerable discussion on the knotty problem of membership fees and how to collect them. It was decided to start charging membership fees from 1994 after legalisation is complete with 3 rates (low: 1 ECU, standard: 20 ECU, high: 50 ECU) (1 ECU equals approximately 1 US dollar). The regional co-ordinators should be responsible for collecting the money and sending it (or part of it) to a general account. The general assembly also appointed new co-ordinators, convenors and standing committee, for details see the names and committees list. The new convenor of the standing committee is Anna Romanowska and the international co-ordinators are Capi Corrales (west), Marketa Novak (central), Inna Berezowskaya and Marie Demlova (east). There will be some joint activity of AWM and EWM at the International Congress of Mathematicians in Zürich, August 1994. This is being organised by Cora Sadosky, the president of AWM, and Eva Bayer. Since the Warsaw meeting it has been decided that the next EWM meeting will be planned to take place in Madrid in July 1995 with Mariemi Alonso (Madrid), Capi Corrales (Madrid) and Rosa Maria Miro (Barcelona) as the main organisers. It is likely that the 1997 meeting can be in Germany.

4. Organising Committees

The Warsaw meeting was organised by the EWM standing committee consisting of Polyna Agranovich (Ukraine), Mariemi Alonso (Spain), Eva Bayer (France), Bodil Branner (Denmark), Jacqueline Detraz (France), Sandra Hayes (Germany), Magdalena Jaroszewska (Poland), Anna Romanowska (Poland), Barbara Roszkowska (Poland), and Caroline Series (England). The local organising committee consisted of Elzbieta Ferenstein, Irmina Herburt, Felicja Okulicka, Ewa Pawelec, Agata Pilitowska, Anna Romanowska, Barbara Roszkowska, and Krystyna Twardowska. The meeting was financially supported by the Technical University of Warsaw, in particular the Dean of the Department of Technical Physics and Applied Mathematics, and by the money left after the third EWM Meeting in Warwick.

MADRID 1995

Organization of EWM

Following the statutes of EWM a general assembly was held during this meeting. The decisions taken are valid until the next general assembly which should take place during the next general meeting of EWM planned for 1997 in ICTP, Trieste.

Decisions taken during the general assembly

Riitta Ulmanen (based on notes taken by her and Karin Bauer)

1 The General Assembly was chaired by Marketa Novak from Sweden who wished everyone present welcome to the General Assembly. The General Assembly was announced in the EWM Newsletter in February 1995 and earlier in separate announcements of the Madrid meeting. Thus the requirements for the announcement of the General Assembly were met and the meeting was valid.

Riitta Ulmanen presented an agenda which was approved as the working procedure of the meeting (Appendix 1)

2 Approving new members

Because this was the first meeting since European Women in Mathematics became an official body everybody would be a new member.

It was decided to approve everyone who had sent her application either to her regional coordinator or to Riitta Ulmanen or who would leave her application form at the General Assembly meeting as a member of the EWM.

2 Electing international coordinators

The following persons were elected as the international coordinators

East: Marie Demlova; Central: Marketa Novak and Inna Berezowskaya;

West: Capi Corrales Rodriganez

3 Confirming regional coordinators

Regional coordinators were confirmed (see XIII).

4 Electing two auditors and a deputy

Seija Kamari and Kirsi Peltonen were elected as auditors for 1996-97. Marja Kankaanrinta was elected to be the deputy auditor. They all are from Finland.

5 Confirming the financial statement and discharging those responsible of liabilities

Marjatta Näätänen explained briefly the financial situation of EWM.

Riitta Ulmanen read aloud the financial statement. It was confirmed by the General Assembly and those responsible of liabilities were discharged.

6 Choosing the place and time for the next meeting

The General Assembly decided to have the next meeting of EWM in 1997. The month was not set yet. The Nordic countries would be responsible for organizing the meeting and it was decided that they choose the place and set the time after that. (It has since been decided that the next meeting is to be held at ICTP, Trieste; the time will be anounced later.)

Also the possibility of Germany organizing the meeting in 1999 was discussed.

7 Electing the Standing Committee and convenor for 1995-97

According to the statutes the Standing Committee consists of 8 - 12 members. The term of a member is four years. Half of the terms will expire at the general assembly meeting and half will continue. After a lively discussion the Standing Committee was elected as follows:

From the Standing Committee for 1993-95 were elected

- Polyna Agranovich, Ukraine
- Bodil Branner, Denmark
- Capi Corrales Rodriganez, Spain
- Marjatta Näätänen, Finland
- Rosa Maria Miro-Roig, Spain
- Caroline Series, United Kingdom

As new members:

- Valentina Barucci, Italy
- Marie Demlova, Check Republic
- Laura Fainsilber, France
- Sylvie Paycha, France
- Ragni Piene, Norway

An election of an additional member was left to the Standing Committee to be made later. (Emilia Mezzetti, Italy, has since been included.)

Sylvie Paycha was elected to be the Convenor and Capi Corrales Rodriganez to be the Deputy Convenor. Marjatta Naatanen was appointed Treasurer.

8 Minutes of the previous General Assembly

EWM became an official body in December 1993 and the previous meeting was in June 1993 in Warsaw. It was decided to accept the brief report made from the Warsaw meeting as the minutes and to approve it.

9 Deciding fees

It was decided to keep the fees as they were: 1 ECU (low), 20 ECU (standard), and 50 ECU (high).

The question of how to collect the fees and how to send it to EWM was raised. Marjatta Näätänen explained that every regional coordinator collects the fees whichever way is most convenient for her. She may open an account for that purpose. After making deductions necessary for local use she then sends the rest to the EWM account in Finland either in her own currency or in Finnish currency.

10 Setting up committees for specific issues

The decisions made appear in the list of committee members at the end of this volume.

THE MATHEMATICAL PART

The mathematical programme constituted the main part of the EWM meeting and similarly the mathematical papers form the main part of these Proceedings.

Included is a description of the philosophy behind the organization of the mathematical part, edited versions of ten lectures given on the three chosen topics (Holomorphic Dynamics, Algebraic Geometry and Mathematical Physics) and of four contributions which formed the basis of an interdisciplinary discussion (on Moduli Spaces), abstracts of three shorter talks and at the end an evaluation of the mathematical aspects of the programme.

The Mathematical part of EWM Meetings

Capi Corrales and Laura Tedeschini Lalli

The organization of the scientific part of an EWM meeting is quite different from that of most mathematical meetings. Starting at the EWM meeting in Luminy in 1991 we decided to experiment with the format trying to reach the following main goals: to learn mathematics which is new to us; to learn how to transmit mathematics; to learn how to discuss mathematics with other mathematicians not necessarily specialists in the same field as we are; and finally to be able to establish scientific links which women, isolated for a number of reasons, can refer to at any stage in their professional career. We have been using the following structure as a model.

1. Before the meeting

Step 1: A scientific committee, chosen by the standing committee of EWM, selects three topics in mathematics. Several considerations are taken into account when choosing the topics:

– the topics should be in the avantgarde of current research;

– the topics should involve beautiful mathematics;

– the topics should try to include also branches of mathematics where, historically, for whatever reasons, the presence of women seems more difficult to detect.

Step 2: Once the topics are chosen, the scientific committee chooses a coordinator for each topic. Several considerations are taken into account when choosing the coordinators:

– their knowledge of the field;

– their commitment to the project of making the transmission of mathematics a main goal of their work;

– their ability and will to work in team with others.

Step 3: The coordinator selects the speakers for her topic. Several considerations are taken into account when choosing the speakers:

– their knowledge of the field;

– their ability, or their will to improve their ability, to transmit knowledge.

Step 4: Coordinators and speaker work together as a team in preparing the talks. The different talks form a whole, and the level of difficulty should be progressive. Once a speaker has been assigned a talk she is invited to give a written draft of her lecture to the coordinator. To ensure crossfield dissemination, and, above all, understandability, the coordinator then distributes these drafts among a few women mathematicians NOT specialists in the topic, who will read them and point out passages where assumptions are taken for granted, or needing an example, or otherwise remaining obscure, etc. We call the crucial function of these professionals "stupid readers", or "naive readers". The coordinator sends the comments of the non-specialists back to the speakers. The speakers make the appropriate corrections and changes and return the text to the coordinators, who send them again to the readers for a final check.

2. The lectures

Many are the questions that frame our work within the mathematical talks. Here are a few of them:

– how do we create an atmosphere in which the audience feels free to ask questions?

– how do we balance the inevitably different levels of knowledge about the topic in a general mathematical audience?

– how do we balance the flow of questions with the flow of the speaker?

– how do we manage to be understood by non-specialists without decaffeinating our expositions?

– mathematics is difficult; how can we make something clear and at the same time keep its richness, depth and not hide its difficulties?

Common sense is a main tool we count on, but we know it is not sufficient. Common sense, patience, and, as scientists, the will and inclination to experiment, try and find by searching. Several strategies have been tested, and as our experience develops, so does the number of strategies that we see work adequately towards answering the above questions. Here are a few:

– one or two women volunteer to concentrate to their fullest ability in the talk and ask questions when they do not follow the speaker, or think this is the case for many in the audience. We label this other crucial function "planted idiots". We think it works best if the planted idiot is actually naive in the field. Other questions are welcome as always;

– the speaker knows ahead of time that when a question is posed by someone in the audience, if someone else knows a more clear or direct way of answering it, this person will speak up. In this way the flow and rhythm of the talk is easier to mantain; and, since the speaker knows this might happen, she does not feel intruded or judged when it does;

– if interdisciplinary connections or other interesting discussions start taking place along the course of a talk, the coordinator of that topic should channel it into organizing a side discussion later, making sure there is a time and a space allowed for it and announced.

3. Writing and publishing the lectures

It is our experience that this is the step where we should be more cautious, since mathematicians have the habit of writing only for specialists. Hence, a process analogue to that of step 4 (before the meeting) is followed: each speaker sends a draft of the text to the coordinator. The coordinator should distribute this draft again among "naive readers", who will make sure the text is faithful to the version and comments agreed on before the actual talk. The coordinator receives the comments of the non-specialists and sends them back to the speaker. The speaker makes the appropriate corrections and returns the new corrected text to the coordinator, who, in turn, sends them again to the readers for a final check.

4. Conclusion

As one can deduce from the above summary, the mathematical part of an EWM meeting is conceived as a learning experience for ALL THE PERSONS taking part in it. Ideally:

– everyone will learn new mathematics, even the specialists. The advantage of speaking clearly to an interdisciplinary audience of mathematicians is that such situation rarely fails to give as fruit the bringing out of connections or points of view thus far unknown to us;

– the speakers will improve their ability as lecturers and mathematical writers;

– everyone will improve her ability to speak about what she works on.

Unfortunately, it is still the case in many European universities that women are singularities within the mathematical departments. Frequently this has a well known inhibiting effect on us, resulting in lack of self-consciousness or defensiveness, both particularly negative when we start our professional path. And if we are inhibited, we do not speak about mathematics, and if we do not speak about mathematics we do not learn how to speak about mathematics, and the loop traps us. The vicious circle of communication, well-known to many, creates a steady isolation which becomes sterile and depressing, as opposed to the temporary isolation which is necessary to all creative work. In fact, we think many problems arise for women in mathematical research from the different types of isolation (communication, life passages...) adding to the second, necessary one, and making it seem unbearable.

5. Other forms: The Interdisciplinary workshops

As we went on planning this EWM meeting we came along words which seem to have different meaning in different branches of mathematics. But often the use of the same words in mathematics points to a common root, a core idea. We think (!) it is one of our original contributions to organize workshops around a word, or an idea, to re-walk paths and rediscover, if not build, common ground on both language and conceptual basis. The first such encounter took place in Madrid, on "Moduli spaces", with speakers from algebraic geometry, number theory, hyperbolic geometry and quantum field theory. In Madrid the next interdisciplinary workshop was put forward, on the words "Renormalization Group". It will hopefully take place in June, 1996 in Paris, with contributions from statistical physics, quantum field theory, markov processes, holomorphic dynamics and real dynamical systems. These workshops are kept more informal, with several persons responsible for illustrating what they deem necessary to the core idea, or the strength of the results that follow in their field. Everybody else is welcome to “pitch in” in workshop style.

6. Poster sessions

Up to now, we have only once experienced a “poster session”. We think it is quite a challenge to our creativity to rethink poster sessions in a way that makes them a good communication tool. We are working on it.

Holomorphic Dynamics

A short course organized by Caroline Series

The aim of the session was to present some of the basic facts and techniques in holomorphic dynamical systems, in particular those defined through iteration in the complex plane of complex polynomials or polynomial-like mappings. All three talks discussed different aspects concerning dynamical behaviours in the dynamical plane as well as properties of families of such maps, represented in a parameter plane.

The first talk given by Bodil Branner was an introduction, emphasizing the classical notion of normal families and its importance in the dynamical plane and the parameter plane (of quadratic polynomials) in relation to Julia sets, and, respectively, the Mandelbrot set.

The second talk given by Núria Fagella focused on polynomial–like mappings and families of such, in particular Mandelbrot–like families, showing the importance of polynomials as local models of more general analytic maps.

The third talk given by Tan Lei discussed two examples of transfer of results from the dynamical planes to the parameter plane, namely asymptotic self–similarities of Julia sets and the Mandelbrot set, and the Hausdorff dimension of certain Julia sets and of the boundary of the Mandelbrot set.

Bodil Branner

Proc. of the 7^{th} EWM meeting, Madrid, 1995

Holomorphic dynamical systems in the complex plane
An introduction

Bodil Branner

Technical University of Denmark, Denmark
branner@mat.dtu.dk

1. A short historic note

The study of *complex analytic dynamics* began at the end of the previous century. The work (by Ernst Schröder, Leopold Leau, Gabriel Koenigs, Lucyan Böttcher and others) was focused on the local behavior of a *complex analytic* function (also called a *holomorphic* function) near a fixed point. With the work of first Arthur Caley and later Pierre Fatou and Gaston Julia the focus changed from local to global behavior.

Fatou and Julia studied – independently of each other – iteration of rational functions. A *rational function* $f(z) = p(z)/q(z)$ is the quotient of two polynomials p and q which are supposed to be relatively prime. The *degree* d of f is defined as the maximum value of the degrees of the polynomials p and q. A rational function can be viewed as a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane, the *Riemann sphere*.

Figure 1: The Riemann sphere.

A rational function is holomorphic, and on the other hand, any holomorphic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational function. If $d \geq 1$ then f is surjective; in fact each point has d preimages (counted with multiplicity). Degree $d = 1$ corresponds to automorphisms of the Riemann sphere, the so called *Möbius transformations*, and degree $d \geq 2$ gives rise to interesting dynamical systems.

Both Fatou and Julia made (1918 – 1920) intensive use of the theory of normal families which had just been introduced and developed by Paul Montel (1912 – 1917) at the time. For a detailed description of the early history, see [A].

Only few papers were published on complex dynamics between 1930 and 1980. But the subject is again a very active area of research. Several new ideas and tools were introduced in the beginning of the eighties. Among them are computer graphics, quasi-conformal mappings (introduced by Dennis Sullivan), polynomial-like mappings and transfer of results from dynamical plane to parameter space (introduced by Adrien Douady and John H. Hubbard).

In this short series of papers we will give examples of the above ideas and tools. This first paper contains basic definitions and results. There are very few proofs. Those which are sketched are chosen to illustrate the concept of normal families and to stress the importance of repelling periodic points.

2. Polynomials

In the rest of the paper we restrict our attention to polynomials of degree $d \geq 2$. Polynomials are exactly those rational functions with the property that $f(\infty) = \infty = f^{-1}(\infty)$. But most often we just think of a polynomial as acting in the complex plane.

To study a polynomial P as a dynamical system means to study the long term behavior for different seeds z_0 of the sequence

$$z_0, z_1 = P(z_0), \ldots, z_n = P(z_{n-1}) = P^n(z_0), \ldots,$$

called the *orbit* of z_0 under iteration. The dynamics take place in the z-plane, the *dynamical plane*.

The goal is both to understand each individual dynamical system for a fixed polynomial P, and to understand how the systems change qualitatively with the polynomial.

In order to understand the dynamics of all polynomials of degree d it is sufficient to consider *monic, centered* polynomials of the form

$$P(z) = z^d + c_{d-2}z^{d-2} + \cdots + c_1 z + c_0;$$

any polynomial F of degree d is namely conjugate to a polynomial of this form through a global affine coordinate change $z \mapsto az + b, a \neq 0$. In other words, the following diagram is commutative

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ {\scriptstyle z\mapsto az+b}\downarrow & & \downarrow{\scriptstyle z\mapsto az+b} \\ \mathbb{C} & \xrightarrow[P]{} & \mathbb{C} \end{array}$$

We identify the set of monic, centered polynomials with the *parameter space* $\mathbb{C}^{d-1} = \{(c_{d-2}, \ldots, c_0)\}$. The polynomials are called centered since the critical points are centered, i.e.

$$\sum_{\omega \in \Omega(P)} \omega = 0$$

where $\Omega(P)$ denotes the set of critical points, i.e. $\Omega(P) = \{z \in \mathbb{C} \mid P'(z) = 0\}$.

The monic, centered quadratic polynomials are of the form

$$Q_c(z) = z^2 + c$$

with one critical point $\omega = 0$. All the examples we give, will be from this family of quadratic polynomials. The family is identified with $\mathbb{C}$, the c-plane, also called the *parameter plane*; it should not be confused with the dynamical plane.

3. The Julia set, the filled Julia set and the Fatou set

For each polynomial P the dynamical plane is decomposed into two complementary sets: the set of points with bounded orbit and the set of points whose orbit tends to ∞. We denote by $K(P)$ the *filled Julia set*, that is the set of seeds with bounded orbit

$$K(P) = \{z \in \mathbb{C} \mid P^n(z) \nrightarrow \infty \text{ as } n \to \infty\}.$$

We denote by $A_P(\infty)$ the set of seeds with orbit tending to ∞:

$$A_P(\infty) = \{z \in \mathbb{C} \mid P^n(z) \to \infty \text{ as } n \to \infty\}.$$

The set $A_P(\infty)$ is called the *attractive basin* of ∞. Both sets are completely invariant under iteration, i.e. under both forward and backward iteration. The common boundary

$$\partial K(P) = \partial A_P(\infty) = J(P)$$

is called the *Julia set*.

Figure 2 shows in black the Julia sets for different quadratic polynomials Q_c.

Example. Consider the simplest quadratic polynomial $Q_0(z) = z^2$. The filled Julia set $K(Q_0)$ equals $\bar{\mathbb{D}}$, the closure of the unit disk; the attractive basin $A_{Q_0}(\infty)$ equals $\mathbb{C} \setminus \bar{\mathbb{D}}$, the exterior of the closed unit disk, and the Julia set $J(Q_0)$ equals S^1, the unit circle.

The above definition of the Julia set is simple, but only valid for polynomials. Many results about Julia sets can only be obtained from the classical and general definition of Julia sets, using the concept of complex analytic normal families. We therefore give this definition as well.

Normal family. Let $U \subset \mathbb{C}$ be an arbitrary domain. A family $\mathcal{F} = \{f_i : U \to \mathbb{C}\}_{i \in I}$ of analytic functions is said to be *normal* if any infinite sequence of functions from $\mathcal{F}$ contain a subsequence that either converges in $\mathbb{C}$ or tends to ∞, uniformly on each compact subset of U.

In complex dynamics we are interested in the families $\mathcal{F}(U) = \{P^n \mid_U : U \to \mathbb{C}\}_{n \geq 0}$ of iterates of the polynomial in question, restricted to arbitrary domains U.

A point $z \in \mathbb{C}$ is said to be *normal* if there exists a neighborhood U of z such that the family $\mathcal{F}(U) = \{P^n \mid_U\}_{n \geq 0}$ is normal.

Definition. The *Fatou set* $F(P)$ is the set of normality, that is $F(P) = \{z \in \mathbb{C} \mid z \text{ normal }\}$, and the *Julia set* $J(P)$ is the complement of $F(P)$ or the set of non-normality. A connected component of the open set $F(P)$ is called a *Fatou component*.

Note that the filled Julia set is the Julia set filled with all the bounded Fatou components.

Example (revisited). Consider again the quadratic polynomial $Q_0(z) = z^2$. The family $\mathcal{F}(\mathbb{D})$ is normal, since each subsequence of the iterates restricted to $\mathbb{D}$ converges to the constant

Figure 2: Julia sets of the polynomials Q_c for different values of c.

function equal to 0, uniformly on any compact set $\bar{D}_r = \{z \in \mathbb{C} \mid |z| \leq r\}$ with $0 < r < 1$. The family $\mathcal{F}(\mathbb{C} \setminus \bar{\mathbb{D}})$ is normal, since each subsequence of the iterates restricted to $\mathbb{C} \setminus \bar{\mathbb{D}}$ converges to the constant function equal to ∞, uniformly on any set $\mathbb{C} \setminus D_R = \{z \in \mathbb{C} \mid |z| \geq R\}$ with $R > 1$. No point on the unit circle is normal. The Fatou set is therefore $\mathbb{C} \setminus S^1$, and the Julia set as before.

Montel proved a useful criterion for normality.

Montel's criterion. Simplest form: Any family of analytic functions defined on a domain U taking values in the plane minus two points, $\mathbb{C} \setminus \{a, b\}$, is normal.
Generalized form: Let $h_j : U \to \mathbb{C}$, $j = 1, 2$, be two analytic functions, satisfying $h_1(z) \neq h_2(z)$ for all $z \in U$. Any family of analytic functions defined on U with values at any $z \in U$ which differ from $h_1(z)$ and $h_2(z)$ is normal.

Note that it follows, that for any neighborhood U of a point in the Julia set the union of orbits which start in U, $\bigcup_{n \geq 0} P^n(U)$, is equal to $\mathbb{C}$ except at most one point. Using this property one can prove that the Julia set is a perfect set, that is a closed set where any point is a limit point in the set; the Julia set has therefore no isolated points.

4. Periodic and preperiodic points

A point z_0 is called *p-periodic* if

$$z_p = z_0 \text{ and } z_j \neq z_0 \text{ for } 0 < j < p;$$

a *fixed point* is a 1-periodic point and a p-periodic point is a fixed point of P^p. A periodic orbit is called a *cycle*. A point z_0 is called *preperiodic* of *preperiod* $k \geq 1$ and *period* p if

$$z_{k+p} = z_k \text{ is a } p\text{-periodic point and } z_j \neq z_k \text{ for } 0 < j < k,$$

see figure 3.

Figure 3: Periodic and preperiodic orbits.

The *multiplier* ρ of a p-periodic point z_0 is defined as the derivative of P^p at z_0. Using the chain rule we obtain

$$\rho = (P^p)'(z_0) = P'(z_{p-1}) \cdot \dots \cdot P'(z_0);$$

the derivative of P^p is therefore the same at all points of the cycle. For this reason ρ is also called the multiplier of the cycle.

We call a cycle

1. *attracting* if $|\rho| < 1$;
2. *superattracting* if $\rho = 0$;
3. *repelling* if $|\rho| > 1$;
4. *indifferent* if $|\rho| = 1$.

Note that a cycle is superattracting if and only if it contains a critical point.

An indifferent cycle has multiplier of the form $\rho = e^{2\pi i\theta}$. The cycle is called *rationally indifferent* or *parabolic* if θ is rational, and *irrationally indifferent* otherwise.

A periodic point z_0 of period p is called *linearizable* if there exists a local holomorphic change of coordinates so that P^p in these coordinates is of the form $\zeta \mapsto \rho\zeta$ where ρ is the multiplier.

Theorem 1 *(Koenigs) An attracting, but not superattracting periodic point is linearizable. A repelling periodic point is linearizable.*

Proof. A sketch. Assume $z_0 = 0$ is an attracting, but not superattracting, fixed point. Since $0 <| \rho |< 1$ there is a neighborhood U of 0 such that $P(U) \subset U$. Set $\varphi_n(z) = P^n(z)/\rho^n$ for $z \in U$. Then

$$\varphi_n(P(z)) = \rho\varphi_{n+1}(z).$$

The holomorphic functions φ_n converge in U, uniformly on compact subsets, to a holomorphic map φ with derivative $\varphi'(z_0) = 1$, defining the required local coordinate change.

For a repelling fixed point we reduce the situation to the above, by considering the branch of P^{-1} which fixes the fixed point.

For periodic points of period p we consider P^p instead of P. ∎

The linearizing coordinates are uniquely determined with the extra requirement: $\varphi'(z_0) = 1$.

Note that it follows from the implicit function theorem that a p-periodic point with multiplier $|\rho| > 1$ can be followed analytically in the parameters in a neighborhood of the polynomial, and in a sufficiently small neighborhood the point remains a p-periodic repelling point. Moreover, the linearizing coordinates vary analytically with the parameters.

Example. Consider again $Q_0(z) = z^2$. The point $z_0 = 1$ is a repelling fixed point with multiplier $\rho = 2$. Set $\varphi(z) = \text{Log } z$, the principal branch of the logarithm defined in the domain $\mathbb{C} \setminus \{z \in \mathbb{C} \mid \text{Re}z \leq 0,\ \text{Im}z = 0\}$. Then φ is the linearizing coordinate change. For $z = re^{2\pi it}, r > 0, -\pi/2 < t < \pi/2$ we have

$$\varphi(Q_0(z)) = \rho\varphi(z) \text{ and } \varphi'(1) = 1.$$

The fixed points for the quadratic polynomials Q_c are solutions to $z^2 + c = z$, hence of the form $z = 1/2 \pm \sqrt{1/4 - c}$. The analytic continuation of the repelling fixed point $z_0 = 1$ is given by $c \mapsto 1/2 + \sqrt{1/4 - c}$ where $\sqrt{\ }$ denotes the principal branch of the root function.

A superattracting periodic point z_0 is of course not linearizable. In the superattracting case there exists a coordinate change so that P^p in these coordinates is of the form $\zeta \mapsto \zeta^k$ where k is the smallest integer n for which the n-th derivative of P^p at z_0 is different from 0. Such coordinates play an important role in the analysis of the local behavior around superattracting orbits; in particular around ∞ which is a superattracting fixed point for all polynomials. For a monic polynomial the coordinates are uniquely determined with the extra requirement $\varphi(z)/z \to 1$ as $z \to \infty$. They are called *Böttcher coordinates* and provide a coordinate change so that P in these coordinates is of the form $\zeta \mapsto \zeta^d$ where d is the degree of the polynomial.

To finish the list: A rationally indifferent periodic point z_0 is not linearizable. An irrationally indifferent periodic point z_0 is called *Siegel* if it is linearizable and *Cremer* if it is not.

A periodic point is contained in the Fatou set if it is attracting or Siegel. All other periodic points are contained in the Julia set. The repelling periodic points are of special interest.

Theorem 2 *The Julia set is the closure of the repelling periodic points.*

The theorem corresponds to Julia's definition of the Julia set, while Fatou used the concept of normal families.

Proof. The proof has two steps:
Step 1. The Julia set is contained in the closure of the periodic points.
Step 2. There are only finitely many non-repelling cycles.

We sketch the first step, and comment on the second step at the end of the next section. Assume the statement is false. Then there exists a point $z_0 \in J(P)$ and a neighborhood U of z_0 without any periodic points. We may assume that z_0 is not a critical value (the image of a critical point) since there are only finitely many critical values for P. Furthermore we may assume that P has a local inverse h_1 defined on U with $h_1(U)$ and U disjoint. Set $h_2(z) = z$. It follows that the family $\mathcal{F}(U) = \{P^n \mid_U\}_{n\geq 0}$ satisfies Montel's normality criterion with respect to the two analytic functions $h_j, j = 1, 2$. This contradicts that z_0 is in $J(P)$ and therefore a non-normal point. ∎

5. Classification of periodic Fatou components

A Fatou component is mapped onto a Fatou component by P. We call a Fatou component V *periodic* if $P^p(V) = V$ for some $p > 0$, *preperiodic* if $P^k(V)$ is periodic for some $k > 0$ and *wandering* otherwise. For a polynomial the attracting basin of ∞ is always a periodic Fatou component of period 1. For a polynomial the different possibilities are listed in the classification theorem below.

Theorem 3 *Let V denote a bounded, periodic Fatou component of period p. Then V is one of the following three types:*

1. **Attracting basin.** *There is a p-periodic attracting point $z_0 \in V$ and all points in V converge to z_0 under iteration of P^p.*

2. **Parabolic basin.** *There is a parabolic point $z_0 \in \partial V$, which is fixed under P^p and satisfies $(P^p)'(z_0) = 1$, and all points in V converge to z_0 under iteration of P^p. (Note that the period p_0 of the periodic point z_0 may be a divisor of p, and the multiplier of z_0 a root of unity which raised to the power p/p_0 equals 1.)*

3. **Siegel disk.** *There is a p-periodic irrationally indifferent point $z_0 \in V$ which is linearizable, the linearizing coordinates are defined on V and the iterate P^p in these coordinates expressed as the irrational rotation $\zeta \mapsto e^{2\pi i\theta}\zeta$ where $\rho = e^{2\pi i\theta}$ is the multiplier.*

In figure 4 we show examples of the three types above. A superattracting basin is also called a *Böttcher domain* and an attracting (but not superattracting) is called a *Schröder domain.* A Parabolic basin is also called a *Leau domain.*

Already Fatou made an exhaustive list of possible types of Fatou components, including the possibility of wandering components. Only the first two possibilities in the classification theorem were known by Fatou to exist, the existence of the third was proved by Carl Siegel in 1942.

The final break through came when Sullivan in 1982 proved the following theorem, using quasi-conformal mappings.

Theorem 4 *(Sullivan) There are no wandering Fatou components for a rational function.*

The non-wandering theorem implies that any Fatou component is either itself periodic of one of the above mentioned three types or eventually mapped onto such a periodic Fatou component.

Relation to critical points. Each type of periodic Fatou components is related to a critical point. Let V be a p-periodic Fatou components as above, and set $\mathcal{V} = \bigcup_{j=0}^{p-1} P^j(V)$. If V is an attracting basin or a parabolic basin then $\mathcal{V}$ contains a critical point (compare with figure 4). If V is a Siegel disk then the boundary of $\mathcal{V}$ is contained in the closure of the orbit of a critical point.

Note that this implies that a polynomial of degree d can have at most $d-1$ cycles which are attracting or parabolic. The statement is in fact also true if we add Cremer and Siegel cycles to the list. (The proof is using the notion of *polynomial-like mappings.*) A polynomial of degree d can therefore have at most $d-1$ non-repelling cycles.

It also follows from the classification theorem that if none of the critical points are attracted to attracting or parabolic cycles and if there are no Siegel disks, then $K(P) = J(P)$. For quadratic polynomials this happens for instance if the critical point is preperiodic. Such a polynomial is called a *Misiurewicz polynomial.* The periodic orbit which the critical point eventually lands on, is always repelling.

6. Hausdorff distance and dependence of Julia sets and filled Julia sets on the polynomial

Both the filled Julia set and the Julia set are non-empty compact sets in the complex plane. The *Hausdorff distance* D_H defines a metric in the set $Comp^*(\mathbb{C})$ of non-empty compact subsets of the complex plane. Given this metric, one can discuss whether the filled Julia set $K(P)$ and the Julia set $J(P)$ depend continuously on the polynomial or not.

Hausdorff distance. Let d denote the Euclidean distance in $\mathbb{C}$, and define for $A, B \in Comp^*(\mathbb{C})$

$$\delta(A,B) = \sup_{a\in A} d(a,B)$$

and

$$D_H(A,B) = \max(\delta(A,B), \delta(B,A)).$$

Figure 4: The three different types of Fatou domains: attracting, parabolic and Siegel.

Observe that it follows from the definition of δ that for any $\epsilon \geq 0$

$$\delta(A, B) \leq \epsilon \iff A \subset B_\epsilon$$

where B_ϵ is the ϵ-neighborhood of B, i.e. $B_\epsilon = \{z \in \mathbb{C} | d(z, B) \leq \epsilon\}$. Moreover, the triangular inequality holds, i.e.

$$\delta(A, C) \leq \delta(A, B) + \delta(B, C) \text{ for all } A, B, C \in Comp^*(\mathbb{C}).$$

With these two properties of δ it follows that D_H is a metric in the space $Comp^*(\mathbb{C})$, in fact a complete metric.

Note that $\delta(A, B_1) \geq \delta(A, B_2)$ if $B_1 \subset B_2$.

The filled Julia set and the Julia set do not in general depend continuously on the polynomial. The general statement is formulated in the following theorem where the polynomials are assumed to be of a fixed degree.

Theorem 5 *(1) The map $P \mapsto K(P)$ satisfies*

$$\delta(K(P), K(P_0)) \to 0 \textit{ when } P \to P_0.$$

(2) The map $P \mapsto J(P)$ satisfies

$$\delta(J(P_0), J(P)) \to 0 \textit{ when } P \to P_0.$$

Recall that we have identified the family of polynomials of degree d with $\mathbb{C}^{d-1}$. That P tends to P_0 therefore means that the coefficients of P tend to the coefficients of P_0.

Corollary 6 *Suppose $K(P_0) = J(P_0)$ for a polynomial P_0. Then both $P \mapsto K(P)$ and $P \mapsto J(P)$ are continuous at P_0.*

Proof. The proof of (1) is more involved that the proof of (2) (see [D]). The proof of (2) is easy and relies essentially on the fact that the repelling periodic points are dense in the Julia set. We sketch the proof of (2).

Fix a polynomial P_0 and an $\epsilon > 0$. Choose a finite number of repelling periodic points in $J(P_0)$, say $X_0 = \{x_1, \ldots, x_N\}$, such that

$$J(P_0) \subset \bigcup_{j=1}^{N} \overline{D(x_j, \frac{\epsilon}{2})},$$

where $D(x_j, \frac{\epsilon}{2})$ denotes the open disk centered at x_j and of radius $\epsilon/2$. We have $\delta(J(P_0), X_0) \leq \epsilon/2$.

Suppose x_j is p_j-periodic, then

$$P_0^{p_j}(x_j) - x_j = 0 \text{ and } |\rho_j| = | (P_0^{p_j})'(x_j) | > 1.$$

It follows from the implicit function theorem that there exists a neighborhood $\mathcal{U}$ of P_0 in the parameter space and analytic functions $P \mapsto \zeta_j(P)$ for $P \in \mathcal{U}$ so that $\zeta_j(P_0) = x_j$ and $\zeta_j(P)$ is a repelling p_j-periodic point. We can assume that $\mathcal{U}$ is chosen so small that $d(x_j, \zeta_j(P)) < \epsilon/2$ for all $P \in \mathcal{U}$. Set $X(P) = \{\zeta_1(P), \ldots, \zeta_N(P)\}$, then $\delta(X_0, X(P)) \leq \epsilon/2$ for all $P \in \mathcal{U}$. Since

$X(P) \subset J(P)$ for all $P \in \mathcal{U}$, it follows that $\delta(X_0, J(P)) \leq \delta(X_0, X(P))$ and therefore that $\delta(J(P_0), J(P)) \leq \epsilon$ for all $P \in \mathcal{U}$. ∎

Note that the Julia set and the filled Julia set are functions which are continuous at any quadratic Misiurewicz polynomial (i.e. a polynomial where the critical point 0 is preperiodic).

One can prove that the Julia set varies dis–continuously at any polynomial with a Siegel cycle, and that both the Julia set and the filled Julia set vary dis–continuously at a polynomial with a parabolic cycle (see again [D]).

7. Parameter space, the Mandelbrot set

The goal is to decompose the parameter space into regions corresponding to qualitatively different dynamical behavior. This is in general a very hard problem. In complex dynamics we divide the parameter space according to qualitatively different behaviors of the finitely many critical points. This turns out to be a good strategy.

We have already seen, that the critical points play an important role in connection with the classification theorem of periodic Fatou components. Another result connected with the critical points is expressed in the following classical theorem, known to Fatou and Julia. Note that this is a global result.

Theorem 7 *(Fatou, Julia) The filled Julia set $K(P)$ is connected if and only if the critical points are contained in $K(P)$.*

The parameter space of monic, centered polynomials is decomposed into two complementary sets: the *connectedness locus* corresponding to polynomials with connected filled Julia set, and the rest, corresponding to polynomials with disconnected filled Julia set.

The Mandelbrot set M is defined as the connectedness locus for the family of quadratic polynomials Q_c

$$M = \{c \in \mathbb{C} \mid K(Q_c) \text{ is connected}\} = \{c \in \mathbb{C} \mid Q_c^n(0) \not\to \infty \text{ as } n \to \infty\},$$

see figure 5.

The first theorem about the Mandelbrot set was the following, proved by Douady and Hubbard in 1981:

Theorem 8 *(Douady, Hubbard) The Mandelbrot set is connected.*

The definition of the Mandelbrot set is very rough, it is surprising that it turns out to give the detailed decomposition of the parameter plane we are interested in. The boundary of the Mandelbrot set is the *bifurcation set*, i.e. the set where the qualitative changes occur. It follows from a theorem of Mañé, Sad, Sullivan that any two polynomials $Q_{c_j}, j = 1, 2$, in the same connected component of $\mathbb{C} \setminus \partial M$ are J-equivalent. That means, there exists a homeomorphism $H : J(Q_{c_1}) \to J(Q_{c_2})$ conjugating the dynamics, i.e. the following diagram is commutative

$$\begin{array}{ccc}
J(Q_{c_1}) & \xrightarrow{Q_{c_1}} & J(Q_{c_1}) \\
{\scriptstyle H}\big\downarrow & & \big\downarrow{\scriptstyle H} \\
J(Q_{c_2}) & \xrightarrow[Q_{c_2}]{} & J(Q_{c_2})
\end{array}$$

Figure 5: The boundary of the Mandelbrot set.

Figure 6 shows two Julia sets that are J-equivalent. The parameters are chosen in M in the same connected component of $\mathbb{C} \setminus \partial M$.

Figure 6: J-stability.

The Mandelbrot set can also be defined within the concept of normal families. Set

$$F_1(c) = c; \quad F_n(c) = (F_{n-1}(c))^2 + c \text{ for } n > 1.$$

For a fixed c the sequence $(0, F_1(c), \ldots, F_n(c), \ldots)$ is just the orbit of the critical point 0 under iteration by Q_c. If $c \notin M$ then $F_n(c) \to \infty$ as $n \to \infty$. If $c \in M$ then $|F_n(c)| \leq 2$ for all n. It follows, that points in ∂M are non–normal for the family $\mathcal{F} = \{F_n\}_{n \geq 0}$, while points in the complement are normal. The boundary of M is therefore the set of non–normality.

This alternative definition can be used to prove the following theorem about Misiurewicz polynomials:

Proposition 9 *Misiurewicz polynomials are dense in the boundary of the Mandelbrot set.*

Proof. Assume the statement is false. Then there exists a point c in ∂M and a simply connected neighborhood U of c without any Misiurewicz polynomials. We may assume that $c \neq 1/4$. Let $g_j : U \to \mathbb{C}$, $j = 1, 2$, denote the two branches of the square-root of $(1/4 - c)$. Then $h_j(c) = 1/2 + g_j(c), j = 1, 2$, determine the two fixed points of Q_c. They differ since $c \neq 1/4$. It follows that the family $\mathcal{F} = \{F_n \mid_U\}_{n \geq 0}$ satisfies Montel's normality criterion with respect to the two analytic functions $h_j, j = 1, 2$. This contradicts that c in ∂M and therefore a non-normal point. ∎

Observe, that we have proved more than stated: the set of Misiurewicz points for which 0 is eventually mapped onto a fixed point is dense in the boundary. The same kind of proof would give that Misiurewicz points for which 0 is eventually mapped onto a periodic orbit of period p is dense in the boundary for any fixed p.

Universality of the Mandelbrot set is discussed in Nuria Fagella's paper. Properties of Misiurewicz polynomials are discussed further in Tan Lei's paper.

References

[A] Daniel S. Alexander, *A History of Complex Dynamics. From Schröder to Fatou and Julia*, Aspect of Mathematics, Vieweg, 1994.

[B] Bodil Branner, *The Mandelbrot set*, in *Chaos and Fractals. The Mathematics Behind the Computer Graphics*, Proc. of Symposia in Appl. Math., AMS, Vol. 39, 1989, pp. 75 – 105.

[D] Adrien Douady, *Does the Julia set depend continuously on the Polynomial ?*, in *Complex Dynamical Systems*, Proc. of Symposia in Appl. Math., AMS, Vol. 49, 1994, pp. 91 – 138.

[DH1] Adrien Douady and John H. Hubbard, *Étude dynamique des polynôme complexes, I, II*, Publications Mathematiques d'Orsay, 1984, 1985.

[DH2] Adrien Douady and John H. Hubbard, *On the dynamics of polynomial–like mappings*, Ann. Ecole Norm. Sup. (4), Vol. 18 (1985), pp. 287 – 343.

[MSS] Ricardo Mañé, Pablo Sad and Dennis Sullivan, *On the dynamics of rational maps*, Ann. Ecole Norm. Sup. (4), Vol. 16 (1983), pp. 193 – 217.

[S] Dennis Sullivan, *Quasi-conformal maps and dynamical systems I. Solutions of the Fatou-Julia problem on wandering domains*, Ann. Math. 122 (1985), pp. 401 – 418.

[St] Norbert Steinmetz, *Rational Iteration, Complex Analytic Dynamical Systems*, de Gruyter Studies in Mathematics, Vol. 16, Walter de Gruyter, 1993.

Proc. of the 7^{th} EWM meeting, Madrid, 1995

The theory of polynomial-like mappings – The importance of quadratic polynomials

Núria Fagella

Univ. Autònoma de Barcelona, Spain
nuria mat.uab.es

1. Introduction

In the field of complex dynamics and, in particular, iteration of functions of one complex variable, the topic that has by far been object of the most attention is the iteration of the family of quadratic polynomials $Q_c := z^2 + c$. In this paper we aim to answer the question of why this very particular family of polynomials is important for the understanding of iteration of general complex functions.

This is the second paper in the "Complex Dynamics" series of EWM 95. We assume that the reader is familiar with the basic definitions and theorems concerning the dynamics of quadratic polynomials which are the topic of the first article [Br3]. For other surveys we refer also [Bl1, Br1] and [Mi].

As a first observation we may say that often, a good place to start is the simplest example, in this case the group of Möbius transformations which are already very well understood. The next simplest class of functions is the class of polynomials of degree two and even that early along the way, we already bump into complicated dynamics which have occupied mathematicians in this field for over twenty years, and still do.

But the real answer to the question has basically one name and that is *the theory of polynomial-like mappings* of A. Douady and J. Hubbard. This theory explains how the understanding of polynomials is not only interesting *per sé* , but helps understand a much wider class of functions namely those that locally behave as polynomials do.

Most of the definitions and results in this paper may be found in the work of Douady and Hubbard *"On the Dynamics of Polynomial-like Mappings"* [DH3]. Our goal is to state their most important results as well as to give several examples that illustrate them. These examples serve also as initial motivation: example B concerns families of cubic polynomials whose dynamical planes exhibit homeomorphic copies of quadratic filled Julia sets (see Figs. 5 and 6), while their parameter spaces contain homeomorphic copies of the Mandelbrot set (see Fig. 12); example C deals with the family of entire transcendental functions $f_\lambda(z) = \lambda\cos(z)$ for which the same phenomena occur (see Figs. 7 and 13); finally, example D shows how we find copies of the Mandelbrot set in the Mandelbrot set itself (see Figs. 8, 9 and 14). Examples of the same phenomena for Newton's method may be found in [BC, CGS, DH3, T] and in [F] for the family $z \mapsto \lambda z e^z$.

This work is divided in two parts, the first one concerning the dynamical planes and the second one the parameter spaces. Section 2 contains the definition of a polynomial-like map and sets up the examples that we follow throughout the paper. In Section 2 we state the straightening theorem (Theorem 2) which explains how polynomial-like maps and actual polynomials are related. Along the way, we give a small survey of the different types of conjugacies that may occur. Section 2 contains the parameter-plane version of the straightening theorem, explaining why we find homeomorphic copies of the Mandelbrot set in the parameter planes of other families of functions.

Figure 12 was borrowed from [Br2] by courtesy of Bodil Branner. All other computer illustrations in this paper were created with the program *It* by Christian Mannes, whom I thank for his assistance and patience.

2. Dynamical Plane

2.1. The Definition of a Polynomial-like Map

Definition A *polynomial-like map* of degree $d \geq 2$ is a triple (f, U', U) where U and U' are open sets of $\mathbb{C}$ isomorphic to discs with $\overline{U'} \subset U$ and $f : U' \longrightarrow U$ is a holomorphic map such that every point in U has exactly d preimages in U' when counted with multiplicity.

Figure 1: The three elements (f, U', U) that form a polynomial-like map.

For the examples throughout the paper the following definition will be necessary.

Definition Let $P(z)$ be a polynomial of degree $d \geq 2$ and let $\varphi(z)$ be the Bötcher coordinates at infinity (see [Br3]). It is a fact that if all critical points of P belong to the filled Julia set $K(P)$ then φ can be extended to map the complement of $K(P)$ to the complement of the unit disk. We define an *equipotential curve of potential* η to be the preimage under φ of a circle of radius e^η. It follows then that an equipotential curve of potential η is mapped under P to an equipotential curve of potential $d\eta$ with degree d.

Example A The obvious example is an actual polynomial of degree d, restricted to a large enough open set. Let P be a polynomial of degree $d \geq 2$ and let Γ' be an equipotential curve of P of some given potential η such that it is a single simple curve. Then, $\Gamma := P(\Gamma')$ is an equipotential curve of potential $d\eta$. If we let U' and U be the open sets enclosed by Γ' and Γ respectively then, the triple $(P|_{U'}, U', U)$ is a polynomial like map (see fig. 2). Note that we do not necessarily have to choose the open sets as regions enclosed by equipotentials. In fact, if we let V' be any large enough disk then $V := P^{-1}(V')$ is an open set contained in V and $(P|_{V'}, V', V)$ is *another* polynomial-like map.

Figure 2: The restriction of two polynomials of degree two as polynomial-like maps. Left: $Q_{-1}(z) = z^2 - 1$ with connected Julia set. Right: $Q_c(z)$ where $c \simeq -0.8 + 0.4i$, with totally disconnected Julia set.

Example B In this example we want to consider some polynomials of degree three which restricted to an open set form a polynomial-like map of degree two. Let P be a cubic polynomial with one critical point ω_1 escaping to infinity under iteration and the other one, ω_2,remaining bounded. Let Γ be the equipotential curve that has the critical value $v_1 := P(\omega_1)$ as one of its points and let U be the open set bounded by Γ. Then, the preimage of Γ under P is a figure eight curve, since all points on Γ have three preimages with the exception of the critical value v_1 that has only two preimages (see fig. 3). This figure eight bounds two connected components. Let U' be the open connected component that contains the critical point ω_2 with a bounded orbit. Then, U' maps to U with degree two, i.e., every point in U has exactly two preimages in U'. The triple $(P|_{U'}, U', U)$ is a polynomial-like map of degree two. (Notice that if we choose sets U' and U as we did in example A, we would obtain a polynomial-like map of degree three.) We have chosen a polynomial of degree three for the sake of the example but it is clear that similar situations would occur with polynomials of any degree, with critical points escaping and not escaping to infinity.

Figure 3: The restriction of a cubic polynomial to create a polynomial-like map of degree two.

Example C Let $f(z) = \pi \cos(z)$ and let U' be the open simply connected domain

$$U' = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < 1.7, |-\pi - \mathrm{Re}(z)| < 2\},$$

and set $U = f(U')$. One can check that $\overline{U'} \subset U$, as shown in Fig. 4. Since U' contains only one critical point $\omega = -\pi$, it follows that f maps U' to U with degree two. Hence the triple $(f|_{U'}, U', U)$ is a polynomial-like of degree two.

Example D Sometimes a polynomial-like map is created as some iterate of a function restricted to a domain. For example, let $Q_c(z) = z^2 + c$ and let $c_0 \backsimeq -1.75778 + 0.0137961i$. Set

$$U' = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < 0.2, |\mathrm{Re}(z)| < 0.2\}.$$

One can check that the polynomial $Q_{c_0}^3$ maps U' onto a larger set U with degree 2, as shown in Fig. 4. The triple $(Q_{c_0}^3|_{U'}, U', U)$ is a polynomial-like map of degree two.

Figure 4: The restriction of $f(z) = \pi\cos(z)$ (left) and $Q_{c_0}^3(z)$ (right) to create polynomial-like maps of degree two.

This is an example of what is called *renormalization*. We say that a quadratic polynomial is *renormalizable* if there exist open disks U' and U and an integer n such that $(f^n|_{U'}, U', U)$ is polynomial like of degree two. Renormalization is a very important topic in the field of complex dynamics. (See [Mc]).

2.2. The Filled Julia Set

The *filled Julia set* and the *Julia* set are defined for polynomial-like maps in the same fashion as for polynomials, keeping in mind that a polynomial-like map is defined only in an open subset of $\mathbb{C}$.

Definition Let $f : U' \longrightarrow U$ be a polynomial-like map. The *filled Julia set* of f is defined as the set of points in U' that never leave U' under iteration, i.e.,

$$K_f := \{z \in U' \mid f^n(z) \in U' \text{ for all } n \geq 0\}.$$

An equivalent definition is

$$K_f = \bigcap_{n \geq 0} f^{-n}(\overline{U'}),$$

and from this expression it is clear that K_f is a compact set.

As for polynomials, we define the *Julia set of f* as

$$J_f := \partial K_f.$$

Notice that if the map f is the restriction of some polynomial F to a set U' then, in general, $K_f \subsetneqq K_F$. As an example consider example B above where F is a polynomial of degree three and f its restriction to the set U' in Fig. 3. Notice that U' maps to U with degree two. The other connected component of $F^{-1}(U)$ which we denote by V, maps to U with degree one. Hence, there are points in U' that map to V and come back to U' afterwards, never leaving the set U. Such points do not belong to K_f since they are not in U' at all times but they belong to K_F since they do not escape to infinity under iteration. Hence $K_f \subsetneq K_F$ and moreover, a connected component C of K_F is either a connected component of K_f or it is disjoint from K_f, since F maps connected components of K_F to connected components. Therefore K_F might have more connected components than K_f but not larger ones.

2.3. The Relation with Polynomials

The Straightenning Theorem stated in this section shows that the relation between polynomial-like maps and actual polynomials is actually very strong. In order to state it, we need to review the different types of equivalences between holomorphic maps.

Equivalences or conjugacies of maps

Suppose $f : U' \longrightarrow U$ and $g : V' \longrightarrow V$ are two polynomials-like maps of degree d. The weakest, but very important equivalence between f and g is what we call *topological equivalence* or *topological conjugacy* and denote by $\sim_{\text{top}}$.

Definition We say that $f \sim_{\text{top}} g$ if there exists φ a homeomorphism from a neighborhood $N(K_f)$ of K_f to a neighborhood $N(K_g)$ of K_g such that the following diagram

$$\begin{array}{ccc} N'(K_f) & \xrightarrow{f} & N(K_f) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ N'(K_g) & \xrightarrow{g} & N(K_g) \end{array}$$

commutes, where $N'(K_f) \subset N(K_f)$ and $N'(K_g) \subset N(K_g)$.

If two functions are topologically conjugate, their dynamics are qualitatively “the same”, since the conjugacy φ must map orbits of f to orbits of g, periodic points of f to periodic points of g, critical points of f to critical points of g, etc. In particular, K_f must be mapped to K_g, but since φ is only a homeomorphism these sets could look quite different. For example, all quadratic polynomials that belong to a given hyperbolic component of the Mandelbrot set (except the center) are topologically equivalent. All polynomials in the complement of the Mandelbrot set are also topologically conjugate. (In fact, these conjugacies are global conjugacies. See remark below.)

On the other hand, the strongest type of equivalence between two holomorphic maps is *conformal equivalence*, due to the rigidity of holomorphic maps.

Definition We say that $f \sim_{\text{conf}} g$ if $f \sim_{\text{top}} g$ and the homeomorphism φ is conformal.

Remark 1 If we were dealing with maps defined in the whole complex plane we could consider also global conjugacies between them. In such a case, if two maps are conformally conjugate then they must be conjugate by an affine map $\varphi(z) = az + b$, since holomorphic isomoprhisms from $\mathbb{C}$ to itself are affine. For the quadratic family, one can easily check that there is a unique representative in each affine class, that is, if Q_{c_1} and Q_{c_2} are affine conjugate, then $c_1 = c_2$.

The concept of *quasi-conformal maps* appears when we want to consider conjugacies that are stronger than topological, but weaker than conformal.

Quasi-conformal mappings For a homeomorphism, we do not have any control whatsoever in how angles are distorted. On the other hand, conformal maps have to preserve angles. Intuitively, a map is quasi-conformal if we have some control on the distortion of angles even if these are not preserved, i.e. the distortion of angles is bounded.

The precise definition is very intuitive if we assume that the map is differentiable. This is not such a crude assumption given the fact that quasi-conformal maps are differentiable almost everywhere. If φ is a diffeomorphism, the tangent map at a given point z_0, takes a certain ellipse in the tangent space at z_0 to a circle in the tangent space at $\varphi(z_0)$. We define the *dilatation of* φ *at* z_0, $\mathcal{D}_\varphi(z_0)$, as the quotient of the length of the major axis over the length of the minor axis of this ellipse.

Definition Let $\varphi : U \to V$ be a diffeomorphism and $\mathcal{D}_\varphi = \sup\limits_{z\in U}\mathcal{D}_\varphi(z)$. Then, φ is K-quasi-conformal if $\mathcal{D}_\varphi \leq K < \infty$.

If we do not assume the map to be differentiable, we can express its distortion in terms of moduli of annuli.

Definition Let φ be a homeomorphism. Then, φ is K-quasi-conformal if for all annuli A in the domain

$$\frac{1}{K}\mathrm{mod}(A) \leq \mathrm{mod}(\varphi(A)) \leq K\mathrm{mod}(A)$$

Note that a map is 1-quasi-conformal if and only if it is conformal.

For those that prefer analytic definitions one can define quasi-conformal maps as follows:

Definition Let φ be a homeomorphism. Then φ is K-quasi-conformal if locally it has distributional derivatives in L^2 and the *complex dilatation* $\mu(z)$ defined locally as

$$\mu(z)\frac{d\overline{z}}{dz} = \frac{\overline{\partial}_z\varphi}{\partial_z\varphi} = \frac{\frac{\partial\varphi}{\partial\overline{z}}}{\frac{\partial\varphi}{\partial z}}\frac{d\overline{z}}{dz}$$

satisfies $|\mu| \leq \frac{K-1}{K+1} := k < 1$ almost everywhere.

For more on quasi-conformal mappings see [A] and [LV].

Quasi-conformal conjugacies and hybrid equivalences We define a *quasi-conformal conjugacy* ($f \sim_{\mathrm{qc}} g$) by requiring the homeomorphism φ in the topological conjugacy to be K-quasi-conformal for some $K \geq 1$. We say that f and g are *hybrid equivalent* ($f \sim_{\mathrm{hb}} g$) if they are quasi-conformally conjugate and the conjugacy φ can be chosen so that $\overline{\partial}_z\varphi = 0$ almost everywhere on K_f. If J_f has measure zero, this simply means that φ is holomorphic in the interior of K_f. Clearly

$$f \sim_{\mathrm{conf}} g \Longrightarrow f \sim_{\mathrm{hb}} g \Longrightarrow f \sim_{\mathrm{qc}} g \Longrightarrow f \sim_{\mathrm{top}} g.$$

The Straightening Theorem

The relation between polynomial-like mappings and actual polynomials is explained in the following theorem, whose proof can be found in [DH3].

Theorem 2 *Let $f : U' \longrightarrow U$ be a polynomial-like map of degree d. Then, f is hybrid equivalent to a polynomial P of degree d. Moreover, if K_f is connected, then P is unique up to (global) conjugation by an affine map.*

This theorem explains why one finds copies of Julia sets of polynomials in the dynamical planes of all kinds of functions. Notice that if f is polynomial-like of degree two and K_f is connected then f is hybrid equivalent to a polynomial of the form $Q_c(z) = z^2 + c$ for a unique value of c by remark 2. This may also be true for other families of polynomial-like maps of degree larger than two, as long as the resulting class of polynomials has a unique representative in each affine class. (As examples, consider the families $\lambda z(1+z/d)^d$, $\lambda \in \mathbb{C}\backslash\{0\}$ for any $d > 2$).

Example B.1 In the setting of example B in Sect.2, we consider the polynomial $P_a(z) = z^3 - 3a^2z - 2a^3 - a$. One can check that for all values of a, the critical point $\omega_2 = -a$ is a fixed point. If we take, for example, $a = -0.6$ then the critical point $\omega_1 = a$ escapes to infinity. By the Straightening Theorem, $P_{-0.6}(z)$ restricted to the open set U' as defined in example B, is hybrid equivalent to a quadratic polynomial and hence, to a polynomial of the form $Q_c(z) = z^2 + c$. In this case, we know that the parameter c must be 0, since $Q_0(z)$ is the only quadratic polynomial of this form with the critical point being fixed. In Fig. 5, we show the dynamical plane of Q_0 and that of $P_{-0.6}$.

Figure 5: Left: the filled Julia set of $Q_0(z) = z^2$ in white. Right: the filled Julia set for $P_{-0.6}(z)$ in white. Note that only the largest component in U' corresponds to the filled Julia set of the polynomial-like map of degree 2.

Example B.2 Again in the setting of example B in sect. 2, we consider the polynomial $R_a(z) = z^3 - 3a^2z + (1/2)(\sqrt{9a^2-4} + a - 4a^3)$. One can check that for all values of a, the critical point $c_2 = -a$ is a point of period 2. In this case we take $a = -0.75$ and then, the critical point $c_1 = a$ escapes to infinity. By the straightening theorem, $R_{-0.75}(z)$ restricted to the open set U' as above, is hybrid equivalent to a quadratic polynomial and hence, to a polynomial of the form $Q_c(z) = z^2 + c$. In this case, we know that the parameter c must be -1, since $Q_{-1}(z)$ is the only quadratic polynomial of this form with the critical point being of period two. In Fig. 6, we show the dynamical plane of $R_{-0.75}$, to be compared with that of Q_{-1} in Fig. 2.

Figure 6: The filled Julia set for $R_{-0.75}$ in white. Note that only the largest component in U' corresponds to the filled Julia set of the polynomial-like map of degree 2. This figure is to be compared with Fig. 2 left.

Example C Even though the function $f(z) = \pi \cos z$ is an entire transcendental function, when restricted to the set U' (as defined in Sect. 2) it is a polynomial-like map of degree two. In Fig. 7, we see in white the set of points that do not escape to infinity (in the imaginary direction) under iteration of f. The largest component inside U' corresponds to the filled Julia set of the polynomial-like map. Since the critical point $-\pi$ is fixed under f, the filled Julia set is homeomorphic to that of $Q_0(z) = z^2$.

Figure 7: The largest white component in U' corresponds to the filled Julia set of $f(z) = \pi \cos z$ restricted to the set U'.

Example D Consider again $Q_{c_0}(z) = z + c_0$ where $c_0 \simeq -1.75778 + 0.0137961$. As explained in Sect. 2, $Q_{c_0}^3$ maps the square box U' centered at 0 and with side length 0.4 onto a larger set U containing U' (see Fig. 4). By the Straightening Theorem, $Q_{c_0}^3$ is hybrid equivalent to

Q_c for some value of c. One can check that the critical point is periodic of period three under iteration of $Q^3_{c_0}$, hence there are a limited number of posibilities for c. In this case the filled Julia set of the polynomial-like map is homeomorphic to the Douady rabbit (see Figs. 8, 9).

Figure 8: The filled Julia set of Q_{c_0}, where $c_0 \simeq -1.76 + 0.01i$.

Figure 9: Left: the Douady rabbit or the filled Julia set of $Q_{c_1}(z) = z^2 - c_1$ in white, where $c_1 \simeq -0.122 + 0.745i$. Right: magnification of the filled Julia set of Q_{c_0} around the critical point. The copy of the Douady rabbit is the filled Julia set of the polynomial-like map corresponding to $Q^3_{c_0}$.

3. Parameter Plane

As usual, the phenomena in dynamical plane are reflected in parameter space. Recall that the parameter space of the family of quadratic polynomials $Q_c(z) = z^2 + c$ contains the *Mandelbrot set* defined as

$$M = \{c \in \mathbb{C} \mid \{Q^n_c(0)\}_{n \geq 0} \text{ is bounded } \}$$

or, equivalently, the set of c values for which the filled Julia set of Q_c is connected (see Fig. 10).

Figure 10: The Mandelbrot set

If we look at the parameter space for other functions, we very often encounter portions that resemble the Mandelbrot set. This fact is again explained by the theory of polynomial-like maps. Since the Mandelbrot set appears when we consider families of quadratic polynomials, it is reasonable to expect that it should also appear when we consider families of polynomial-like maps of degree two, as long as these families are "nice" enough.

Remark 3 For the sake of exposition, we consider here only one parameter families of polynomial-like mappings of degree two. For other cases see [DH3].

3.1. Analytic families of polynomial-like mappings

Definition Let Λ be a Riemann surface and $\mathcal{F} = \{f_\lambda : U'_\lambda \to U_\lambda\}$ be a family of polynomial-like mappings. Set

$$\begin{aligned} \mathcal{U} &= \{(\lambda, z) \mid z \in U_\lambda\} \\ \mathcal{U}' &= \{(\lambda, z) \mid z \in U'_\lambda\} \\ f(\lambda, z) &= (\lambda, f_\lambda(z)) \end{aligned}$$

Then, $\mathcal{F}$ is an analytic family of polynomial-like maps if it satisfies the following properties:

1. $\mathcal{U}$ and $\mathcal{U}'$ are homeomorphic over Λ to $\Lambda \times \mathbb{D}$
2. The projection from the closure of $\mathcal{U}'$ in $\mathcal{U}$ to Λ is proper
3. The map $f : \mathcal{U}' \to \mathcal{U}$ is holomorphic and proper

If these properties are satisfied, the degree of the maps is constant and it is called the degree of $\mathcal{F}$. We denote $K_\lambda = K_{f_\lambda}$ and $J_\lambda = J_{f_\lambda}$. By the Straightening Theorem, for each λ the map f_λ is hybrid equivalent to a polynomial of degree the degree of $\mathcal{F}$. By analogy with polynomials, we define

$$M_{\mathcal{F}} = \{\lambda \in \Lambda \mid K_\lambda \text{ is connected } \}.$$

In the next section, we give some conditions under which the set $M_{\mathcal{F}}$ is homeomorphic to the Mandelbrot set.

3.2. Homeomorphic Copies of the Mandelbrot Set

Let $\mathcal{F}$ be an analytic family of polynomial-like maps of degree two. Then, for each $\lambda \in M_{\mathcal{F}}$, f_λ is hybrid equivalent to a unique polynomial of the form $Q_c(z) = z^2 + c$. Hence the map

$$\begin{array}{rccc} \mathcal{C}: & M_{\mathcal{F}} & \longrightarrow & M \\ & \lambda & \longmapsto & c = \mathcal{C}(\lambda) \end{array}$$

is well defined.

Theorem 4 *Let $A \in \Lambda$ be a closed set of parameters homeomorphic to a disc and containing $M_{\mathcal{F}}$. Let ω_λ be the critical point of f_λ and suppose that for each $\lambda \in \Lambda \setminus A$, the critical value $f_\lambda(\omega_\lambda) \in U_\lambda \setminus U_{\lambda'}$. Assume also that as λ goes once around ∂A, the vector $f_\lambda(\omega_\lambda) - \omega_\lambda$ turns once around* 0 *(see Fig. 11). Then, the map $\mathcal{C}$ is a homeomorphism and it is analytic in the interior of $M_{\mathcal{F}}$.*

Figure 11: Illustration of theorem 4.

Remarks 5

1. The assumption "$f_\lambda(\omega_\lambda) \in U_\lambda \setminus U_{\lambda'}$ if $\lambda \in \Lambda \setminus A$" is equivalent to $M_{\mathcal{F}}$ being compact.
2. If the winding number of $f_\lambda(\omega_\lambda) - \omega_\lambda$ around 0 is $\delta > 1$, then $\mathcal{C}$ is a branched covering of degree δ.

Example A The purpose of this example is to illustrate that the conditions of the theorem are satisfied for the Mandelbrot set itself. Consider the parameter plane for the quadratic family and let

$$\Lambda = \{c \mid G_M(c) < 2\eta\} A = \{c \mid G_M(c) \leq \eta\}$$

where G_M denotes the Green's function of the Mandelbrot set. Given the way the Green's function of M is defined, if $c \in \partial A$ then c lies on an equipotential curve of potential η in the dynamical plane as well. So, for each $c \in \partial A$, let Γ'_c and Γ_c be the equipotential curves in the dynamical plane of Q_c of potentials η and 2η respectively. The open sets enclosed by Γ'_c and Γ_c are the discs U'_c and U_c respectively and $\mathcal{F} = (Q_c|_{U'_c}, U'_c, U_c)$ the analytic family of polynomial-like maps. Note that, by construction, for each $c \in \Lambda \setminus A$, the critical value $Q_c(0) = c$ lies in $U_c \setminus U'_c$. Also, as c turns once around ∂A, the critical value c turns once around the critical point 0. In this case $M_{\mathcal{F}} = M$.

Example B Consider the family of cubic polynomials $P(z) = P_{a,b}(z) = z^3 + az + b$. For any given constants ρ and θ we define the parameter space $\Lambda_\theta = \Lambda_{\rho,\theta}$ to be the set of polynomials P such that:

- one critical point ω_1 escapes to infinity with escape rate ρ
- another critical point ω_2 escapes to infinity at a slower rate or stays bounded
- the *co-critical point* ω_1' of ω_1 that is, the other preimage of $P(\omega_1)$ different from ω_1, belongs to the external ray $R(\theta)$ (see [Br3] for definitions of this terms and [Br2] for more in this example).

Note that polynomials of this type are polynomial-like maps of degree two, as shown in example B in Sect. 2. In [BH] Branner and Hubbard prove:

Theorem 6 *The parameter space Λ_θ is homeomorphic to a disc.*

Hence, polynomials in Λ_θ form a one-parameter family of polynomial-like maps of degree two.

Let $B_\theta = B_{\rho,\theta}$ be the set of polynomials in Λ_θ for which the orbit of ω_2 is bounded. Note that examples B.1 and B.2 are in B_θ for some values of ρ and θ. Also in [BH] we find the following theorem:

Theorem 7 *Let $\lambda \in B_\theta$ and suppose that the connected component of c_2 in $K(P_\lambda)$ is periodic. Then, the connected component of λ in B_θ is a homeomorphic copy of the Mandelbrot set.*

Figure 12 shows the parameter space Λ_0 with B_0 in black.

Figure 12: The set $B_0 \subset \Lambda_0$ shown in black, with countably many components which are homeomorphic copies of the Mandelbrot set.

Example C Let $f_\lambda(z) = \lambda \cos(z)$ and let A be an appropriately chosen disc in the λ-plane around $\lambda = \pi$. One can check that for appropriate choices of U_λ' and U_λ, the maps $(f_\lambda|_{U_\lambda'}, U_\lambda', U_\lambda)$ form an analytic family of polynomial-like maps. As λ turns once around ∂A, the critical point stays fixed at $-\pi$ while the critical value $-\lambda$ winds once around $-\pi$ hence

Figure 13: Left: Parameter plane of $f_\lambda(z) = \lambda \cos z$. Right: magnification of the copy of the Mandelbrot set centered around $\lambda = \pi$.

satisfying the conditions of theorem 4. In Fig. 13 we see the resulting copy of the Mandelbrot set, with $\lambda = \pi$ as the center of its main cardioid.

Example D

Let $A \subset \Lambda$ be a small discs of parameters centered at $c \backsimeq -1.755$ and with c_0 contained in A where c_0 is as in example D in Sect. 2. For Q_{c_*}, the critical point is periodic of period three. One can check that for apropiate choices of λ, U_c, U'_c and A, the conditions of the theorem are satisfied for the family $\mathcal{F} = \{Q_c^3 : U'_c \to U_c\}_{c \in \Lambda}$. Figure 14 shows the Mandelbrot set and a magnification of the homeomorphic copy that contains c_0.

Figure 14: Copy of the Mandelbrot set in the parameter plane of Q_c. Range:$[-1.8, -1.72] \times [-0.038, 0.038]$.

References

[A] L. Ahlfors, Lectures on Quasi-conformal Mappings, Van Nostrand, 1966.

[Bl1] P. Blanchard, Complex Analytic Dynamics on the Riemann Sphere, *Bull. Amer. Math. Soc.* **11** (1984), 85-141.

[Bl2] P. Blanchard, The Dynamics of Newton's Method, *Proc. Symp. Applied. Math.*, **49** (1994) 139-154.

[BC] P. Blanchard & A. Chiu, Complex Dynamics: An Informal Discussion. Book chapter in Fractal Geometry and Analysis, ed. J. Bélair and S. Dubuc, Kluwer Academic Publishers (1991) 45-98.

[Br1] B. Branner, The Mandelbrot Set, *Proc. Symp. Applied Math.*, (1989), 75-105.

[Br2] B. Branner, Puzzles and para-puzzles of quadratic and cubic polynomials, *Proc. Symp. Applied. Math.*, **49** (1994) 31-69.

[Br3] B. Branner, Holomorphic Dynamical Systems in the Plane – An introduction, EWM95.

[BH] B. Branner & J. H. Hubbard, The Iteration of Cubic Polynomials. Part I: The Global Topology of Parameter Space, *Acta Mathematica*, **160**, (1988), 143-206.

[CGS] J. Curry, L. Garnett & D. Sullivan, On the iteration of a rational function: Computer experiment with Newton's method, *Comm. Math. Phys* **91** (1983), 267-277.

[DH1] A.Douady & J.Hubbard, Iterations des Polynomes Quadratiques Complexes, *C.R. Acad. Sci., Paris, Ser.I* **29** (1982), 123-126.

[DH2] A.Douady & J.Hubbard, Etude dynamique des polynomes complexes, I,II, *Publ. Math. Orsay* (1984, 1985).

[DH3] A. Douady & J.Hubbard, On the Dynamics of Polynomial-like Mappings, *Ann. Scient., Ec. Norm. Sup.* 4^e *series*, **18** (1985) 287-343.

[F] N. Fagella, Limiting Dynamics of the Complex Standard Family, *Int. J. of Bif. and Chaos* (1995), To appear.

[LV] O. Lehto & K. J. Virtanen, Quasi-conformal Mappings in the Plane, Springer-Verlag, 1973.

[Mc] C. T. McMullen, Complex Dynamics and Renormalization, *Annals of Mathematics Studies*, 135, Princeton University Press, 1994.

[Mi] J. Milnor, Dynamics on One Complex Variable: Introductory Lectures, IMS at Stony Brook Preprint n. 1990/5 (1992).

[MSS] R. Mañe, P. Sad & D. Sullivan, On the Dynamics of Rational Maps, *Ann. Scie. École Norm. Sup.* (4) **16** (1983), 193-217.

[T] Tan Lei, Cubic Newton's method of Thurston's type, preprint, Laboratoire de Math., École Norm. Sup. de Lyon.

Proc. of the 7^{th} EWM meeting, Madrid, 1995

Local properties of the Mandelbrot set M Similarities between M and Julia sets

Tan Lei

University of Warwick, England
tanlei@maths.warwick.ac.uk

1. Introduction

Local properties of the Mandelbrot set around a point c_0, such as self similarity, Hausdorff dimension, local connectivity, are closely related to properties of the filled Julia set K_c for c in a neighborhood of c_0. Recall that $M = \{c \in \mathbb{C} \mid 0 \in K_c\}$. For each $N \geq 0$, let F_N denote the holomorphic mapping $c \mapsto Q_c^N(0)$, where Q_c denotes the polynomial $z \mapsto z^2 + c$. Since K_c is totally invariant, for any $N \in \mathbb{N}$, we have

$$M = \{c \in \mathbb{C} \mid F_N(c) \in K_c\} \ .$$

It is often convenient to go to the product space $(c, z) \in \mathbb{C} \times \mathbb{C}$ to study both M and K_c. We may regard $c \mapsto K_c$ as a map and $\mathcal{K} = \{(c, z) \mid c \in \mathbb{C},\ z \in K_c\}$ as its graph in $\mathbb{C} \times \mathbb{C}$. Then M can be interpreted as

$$Proj_c\left(graph(F_N) \bigcap \mathcal{K}\right) \tag{1}$$

where $Proj_c$ denotes the projection mapping to the first coordinate (the c coordinate).

Let c_0 be a point of M, so $F_N(c_0) \in K_{c_0}$. A typical way to relate K_{c_0} to the local structure of M around c_0 is to study the local structure of $\mathcal{K}$ around the point $(c_0, F_N(c_0))$ (which depends on the regularity of the mapping $c \mapsto K_c$ at c_0, see for example conditions * and *' below), and the slope of F_N at c_0 (i.e. the constant $F_N'(c_0)$). In this paper we illustrate this technique by showing two related results.

The first theorem states a similarity result between the dynamical plane and the parameter plane around Misiurewicz points. This similarity can actually be observed in computer experiments as in Figures 1, 2 and 3. We know that the set of such points form a dense subset of ∂M, and for each Misiurewicz point c_0, we have $J_{c_0} = K_{c_0}$ (see B. Branner's paper).

Denote by D_H the Hausdorff distance on the space $Comp^*(\mathbb{C})$ of non-empty compact subsets of $\mathbb{C}$. Let τ_{-c} denote the translation $z \mapsto z - c$. For any closed set $A \subset \mathbb{C}$, define

$$A_r = (A \cap \overline{\Delta}(0, r)) \cup \partial\Delta(0, r)$$

where $\Delta(z, r)$ denotes the open disc centered at z and with radius r. For technical reasons it is important to include the circle $\partial\Delta(0, r)$ when measuring the Hausdorff distance of two closed sets within the disc $\overline{\Delta}(0, r)$.

Figure 1: Enlargement of the Mandelbrot set around the Missurewicz point $c_0 \sim -0.77568377 + 0.13646737i$ to be compared with Fig. 2.

Let ρ be a complex number with $|\rho| > 1$. A closed set $A \subset \mathbb{C}$ is said to be *ρ-self-similar* about x if $\rho\tau_{-x}(A) = \tau_{-x}(A)$; it is said to be *asymptotically ρ-self-similar* about x if there is a ρ-self-similar set L (about 0) so that the Hausdorff distance $D_H((\rho^n\tau_{-x}(A))_r, L_r)$ tends to zero as n tends to infinity for some $r > 0$ (hence every $r > 0$). For an example of a self-similar set, see the appendix.

Theorem 1 *([T1]): For every Misiurewicz point c_0, there are two constants $\rho \in \mathbb{C}, |\rho| > 1$ and $\mu \in \mathbb{C} - \{0\}$, and a closed ρ-self-similar set $L \subset \mathbb{C}$ such that for any $r > 0$*

a) *$D_H((\rho^n\tau_{-c_0}(K_{c_0}))_r, L_r) \to 0$ as $n \to \infty$, i.e. K_{c_0} is asymptotically ρ-self-similar about c_0. Moreover, for c in a neighborhood of c_0, we have $D_H((\rho(c)^n\tau_{-\zeta(c)}(K_c))_r, L(c)_r) \to 0$, where $c \mapsto \rho(c)$ and $c \mapsto \zeta(c)$ are holomorphic, and $c \mapsto L(c)$ is a map verifying the condition * below.*

b) *$D_H((\rho^n\tau_{-c_0}(M))_r, (\mu L)_r) \to 0$, i.e. M is also asymptotically ρ-self-similar about c_0.*

c) *$\lim_{t\in\mathbb{C},|t|\to\infty} D_H((t\mu\cdot\tau_{-c_0}(K_{c_0}))_r, (t\tau_{-c_0}(M))_r) = 0$. Hence up to multiplication by μ the sets K_{c_0} and M are asymptotically similar about c_0.*

(We will give the precise form of ρ, L and μ later in the formulas (2) , (6)).

Part c) is an easy consequence of a) and b). As an application of this result, one can show that M is locally connected at each Misiurewicz point (see the appendix).

Figure 2: The Julia set of the Missurewicz polynomial $z \mapsto z^2 + c_0$, where c_0 is as in Fig. 1.

Figure 3: Enlargment of Fig. 2 around the point c_0.

The second result can be considered as a quantitative study of the similarity between Julia sets and the Mandelbrot set. It gives estimates for the Hausdorff dimension of these sets. The exact definition of the Hausdorff dimension is not so important for the purpose of this paper. It can be found in the appendix. We only state two basic properties of it: For any compact set K of $\mathbb{C}$, we have H-dim$(K) \in [0,2]$, and for any compact subset $K' \subset K$, we have H-dim$(K') \leq$ H-dim(K). One should consider that the Hausdorff dimension measures a kind of density or complexity of a set. So if $K \subset \mathbb{C}$ is a compact set without interior, but with H-dim$(K) = 2$, then K must be very complicated.

Theorem 2 *(Shishikura): For each $\varepsilon > 0$, there is a dense subset of ∂M satisfying that for every point c_0 in this set, there is a closed set $X \subset \partial K_{c_0}$ and a constant $r_0 > 0$ such that*

a') H-dim$(\partial K_{c_0}) \geq$ H-dim$(X) > 2 - \varepsilon$. *Moreover for $c \in \Delta(c_0, r_0)$, we have similarly* H-dim$(X(c)) > 2 - \varepsilon$, *where $X(c)$ is a subset of ∂K_c, and $c \mapsto X(c)$ verifies the condition *' below.*

b') H-dim$(\partial M \cap \Delta(c_0, r_0)) > 2 - \varepsilon$.

Corollary. We have H-dim$(\partial M) = 2$.

The existence of c_0 and X satisfying a') involves deep analysis of parabolic perturbations and renormalizations. We will give some ideas of it in the appendix. The set X is in fact a hyperbolic set (see the appendix). It is this hyperbolicity that guarantees the stability property.

We will sketch the proofs of a),b),b') in the following sections.

Condition *. Consider a mapping $c \mapsto A(c)$ with $c \in \Delta(c_0, r_0)$, $A(c)$ a closed subset of $\mathbb{C}$, such that $c \mapsto A(c)_r$ is continuous at c_0, for every $r > 0$. The mapping *admits a dense set of continuous sections at* c_0, if there exists a dense subset $Z \subset A(c_0)$ and, for each $z \in Z$, a neighborhood $U_z \subset \Delta(c_0, r_0)$ of c_0, and a mapping

$$s : \{(c, z) \mid z \in Z,\ c \in U_z\} \to \mathbb{C}$$

such that $s(c_0, \cdot) = id$ and $s(\cdot, z) : U_z \to \mathbb{C}$ is continuous with $s(c, z) \in A(c)$.

Condition *'. Consider a mapping $c \mapsto X(c)$, with $c \in \Delta(c_0, r_0)$, $X(c)$ a subset of $\mathbb{C}$. The mapping admits a *holomorphic motion* if there is a mapping $i : \Delta(c_0, r_0) \times X \to \mathbb{C}$, where $X = X(c_0)$, such that $i(c_0, \cdot) = id$, $i(c, \cdot) : X \to \overline{\mathbb{C}}$ is injective with $i(c, X) = X(c)$ and $i(\cdot, z) : \Delta(c_0, r_0) \to \overline{\mathbb{C}}$ is holomorphic for each $z \in X$.

2. Dynamical planes, Proof of a)

First remark that asymptotic similarity is invariant under conformal transformations. More precisely:

Proposition 3 *: Let U and V be neighborhoods of* 0 *and $u : U \to V$ be an injective analytic map satisfying $u(0) = 0$ and $u'(0) \neq 0$. Suppose A is a closed subset of U containing* 0, *and suppose $u(A)$ is asymptotically ρ-self-similar for some $\rho \in \mathbb{C}$ with $|\rho| > 1$, i.e. for any $r > 0$,*

$$D_H((\rho^n u(A))_r, L_r) \to 0 \quad as \quad n \to \infty$$

where L is a closed ρ-self-similar set. Then A is asymptotically ρ-self-similar to $(1/u'(0))L$, i.e. for any $r > 0$,

$$D_H\left((\rho^n A)_r, \left(\frac{1}{u'(0)}L\right)_r\right) \to 0 \ .$$

Now assume that c_0 is a Misiurewicz point. If there is no ambiguity, we simplify the notation by setting $Q_{c_0} = Q$ and $K_{c_0} = K$. By definition, there is a smallest number k such that $\alpha = Q^k(c_0)$ is a periodic point. Let $\rho = (Q^p)'(\alpha)$ denote the multiplier.

It follows from classical results of Fatou and Julia that the point α is repelling, i.e. $|\rho| > 1$. Let $\varphi : U \to \Delta(0,r)$ denote linearizing coordinates in a neighborhood U of α, hence satisfying $\varphi(\alpha) = 0$, $\varphi'(\alpha) = 1$ and $\varphi \circ Q^p \circ \varphi^{-1}(z) = \rho z$ for all $z \in \Delta(0, r/|\rho|)$, for some $r > 0$.

Since K is totally invariant $(Q(K) = Q^{-1}(K) = K)$ and φ is a linearizing coordinate we have

$$(\rho\varphi(K \cap U))_r = (\varphi(K \cap U))_r \ .$$

Applying proposition 3 to $(u, A) = (\varphi, K \cap \overline{U})$ we get

$$D_H((\rho^n \tau_{-\alpha}(K))_{r'}, (\varphi(K \cap \overline{U}))_{r'}) \to 0$$

for any $0 < r' < r$.

Since $(Q^k)'(c_0) \neq 0$, there exists a neighborhood V of c_0 and $r'' > 0$ such that $\varphi \circ Q^k : V \to \Delta(0, r'')$ is a homeomorphism. Applying proposition 3 to $(u, A) = (\varphi \circ Q^k, K \cap \overline{V})$ we obtain

$$D_H\left((\rho^n \tau_{-c_0}(K))_{r''}, \left(\frac{1}{(Q^k)'(c_0)}\varphi(K \cap \overline{U})\right)_{r''}\right) \to 0 \ .$$

This proves the first assertion of a). Note that ρ is the multiplier of the periodic point α and that the ρ-self-similar limit set L is determined (locally) by

$$\frac{1}{(Q^k)'(c_0)}\varphi \circ Q^k(K \cap \overline{V}) \tag{2}$$

where V is any neighborhood of c_0 which is mapped homeomorphically onto its image under $\varphi \circ Q^k$.

Example. Let us take the example of $c_0 = i$. For $Q_i : z \mapsto z^2 + i$, the orbit of i is : $i \mapsto i - 1 \mapsto -i \mapsto i - 1$. In our notation, $k = 1, p = 2, \alpha = i - 1$ and $\rho = (Q_i^2)'(i-1) = 4(1+i) = 4\sqrt{2}e^{\pi i/4}$.

Remark. Note that a) could have been stated in greater generality. The statement is true for any repelling periodic point α with multiplier ρ and linearizing coordinates φ, and similarly for any pre-periodic repelling point. We have only chosen the special pre-periodic point c_0 in order to be able to compare with the parameter plane. These are the only properties we have used, together with the invariance of K.

The second assertion of a) is a consequence of a stability result. As before let c_0 be a Misiurewicz point and let k, p, α, ρ be as above for the map Q_{c_0}.

As an application of the implicit function theorem, (pre-)repelling periodic points are "stable" with respect to the parameter. That is for any c in a neighborhood W of c_0, the polynomial $Q_c : z \mapsto z^2 + c$ has a p-periodic point $\alpha(c)$, depending analytically on c, with multiplier $\rho(c)$, and with a k-th pre-image $\zeta(c)$ of $\alpha(c)$, both depending analytically on c, satisfying

$$\zeta(c_0) = c_0 \ , \ \alpha(c_0) = \alpha \ , \ \rho(c_0) = \rho \ \text{ and } (Q_c^k)'(\zeta(c)) \neq 0 \ .$$

Let φ_c denote the linearization coordinate around $\alpha(c)$. The same proof as above shows that there is a neighborhood $V_{\zeta(c)}$ of $\zeta(c)$ which is mapped by $\varphi_c \circ Q_c^k$ comformally onto its image, and that K_c is asymptotically $\rho(c)$-self-similar about $\zeta(c)$ to the limit set (locally)

$$L(c) = \frac{1}{(Q_c^k)'(z)|_{z=\zeta(c)}} \varphi_c \circ Q_c^k(K_c \cap \overline{V}_{\zeta(c)}) \ . \tag{3}$$

As for the condition *: the mapping $c \mapsto L(c)$ is continuous at c_0 because $c \mapsto K_c$ is continuous at c_0 (Douady-Hubbard, see B. Branner's paper); each repelling periodic point has a continuous section, and the set of repelling points is dense in $J_{c_0} = K_{c_0}$.

3. Parameter plane, Proofs of b),b')

The proof of b) is done in two steps, one consists of a general result, one is the adaptation.

Proposition 4 *Suppose Λ is a neighborhood of λ_0 in $\mathbb{C}$. Assume we have a mapping $\lambda \mapsto A(\lambda)$ satisfying the condition * at λ_0, and that $A(\lambda)$ is $\rho(\lambda)$-self-similar about 0, where $\lambda \mapsto \rho(\lambda)$ is holomorphic with $|\rho(\lambda_0)| > 1$. Assume $u : \Lambda \to \mathbb{C}$ is a holomorphic mapping, with $u(\lambda_0) = 0$, and $u'(\lambda_0) \neq 0$ (transversality). Set*

$$M_u = \{\lambda \in \Lambda | u(\lambda) \in A(\lambda)\} \tag{4}$$

Then M_u is asymptotically $\rho(\lambda_0)$-self-similar about λ_0 to the $\rho(\lambda_0)$- self-similar set $A(\lambda_0)/u'(\lambda_0)$.

Proof. (sketch) Assume $\lambda_0 = 0$. For $z \in \mathbb{C}$ a point and $K \subset \mathbb{C}$ a compact set, we use $d(z, K)$ to denote the euclidean distance from z to K, i.e. $d(z, K) = \min_{z' \in K} |z - z'|$.

To fix our ideas we treat first two simple cases. Assume that $\lambda \mapsto A(\lambda)$ is a constant map (i.e. $A(\lambda) \equiv A(\lambda_0)$) and $u(z) = u'(0)z$ is linear. Then obviously M_u is ρ_0-self-similar about 0, to the set $A(0)/u'(0)$. Assume now that $\lambda \mapsto A(\lambda)$ is still constant but $u(z)$ is no longer linear. Then M_u coincides with $u^{-1}(A(0))$ and the conclusion holds by Proposition 3.

Now let us come back to the setting of our proposition. Set $\rho(0) = \rho$, $u'(0) = u'$ and $A(0) = A$. Choose $r > 0$ sufficiently small. We must prove that $D_H((\rho^n u' M_u)_r, A_r) \to 0$ as $n \to \infty$. Recall that by definition $D_H(A, B) = \max(\delta(A, B), \delta(B, A))$ (see B. Branner's paper).

First we prove $\delta((\rho^n u' M_u)_r, A_r) \to 0$. Let

$$\lambda \in M_u \cap \overline{\Delta}(0, \frac{r}{\rho^n u'})$$

so that $\rho^n u' \lambda \in \overline{\Delta}(0, r)$ and $|\lambda| \sim |\rho|^{-n}$.

$$d(\rho^n u' \lambda, A_r) \leq |\rho^n u' \lambda - \rho^n u(\lambda)| + |\rho^n u(\lambda) - \rho(\lambda)^n u(\lambda)| + d(\rho(\lambda)^n u(\lambda), A_r) = I_1 + I_2 + I_3 \ .$$

We have $I_1 \sim |\rho|^n |\lambda|^2 \sim |\rho|^n |\rho|^{-2n} \to 0$, and

$$I_2 \ = \ |\rho^n - \rho(\lambda)^n| \cdot |u(\lambda)| \ \leq \ \sum_{i=0}^{n-1} |\rho|^{n-i-1} |\rho(\lambda)|^i |\rho - \rho(\lambda)| \cdot |u(\lambda)| \ \sim \ n|\rho|^{n-1}|\lambda|^2 \ \to \ 0 \ .$$

$I_3 \to 0$ because $\lambda \in M_u$ so $\rho(\lambda)^n u(\lambda) \in A(\lambda)$ and $|\rho(\lambda)^n u(\lambda)| \sim |\rho^n u' \lambda| \leq r$, moreover $\delta(A(\lambda)_r, A_r) \to 0$ as $\lambda \to 0$.

To prove $\delta(A_r, (\rho^n u' M_u)_r) \to 0$, we only need to show $d(z, (\rho^n u' M_u)_r) \to 0$ for each $z \in Z$, where Z is the dense set in A_r where we have continuous sections $s(\cdot, z)$ (see condition *). Let $z \in Z$. By Rouché's theorem, for n large, there is a $\lambda_n \in U_z$ as a solution of the equation $s(\lambda, z) = \rho(\lambda)^n u(\lambda)$. Hence $\lambda_n \in M_u$ and $|\lambda_n| \leq C|u(\lambda_n)| \sim |\rho|^{-n}$. Moreover

$$|\rho(\lambda_n)^n u(\lambda_n)| = |s(\lambda_n, z)| \sim |s(0, z)| = |z| \leq r \ .$$

So $d(z, (\rho^n u' M_u)_r) \ \leq \ |z - \rho^n u' \lambda_n| \ \leq \ |z - s(\lambda_n, z)| + |\rho(\lambda_n)^n u(\lambda_n) - \rho^n u' \lambda_n| \ = \ I_4 + I_5$.

We have $I_5 \to 0$ as above, and $I_4 \to 0$ by the continuity of $s(\cdot, z)$, and the fact $s(0, z) = z$. ∎

Remark. This proposition is also true in a higher dimensional setting, and under a weaker hypothesis of differentiability for ρ and u. See [T1] for details.

Let c_0 be a Misiurewicz point. We now will adapt the situation just considered. Recall from the proof of the second part of a) that there exist a neighborhood W of c_0 and a holomorphic mapping $\zeta(c)$ such that K_c is asymptotically $\rho(c)$-self-similar about the point $\zeta(c)$, to the $\rho(c)$-self-similar limit set $L(c)$ (see (3)). Shrinking W if necessary we have $c \in V_{\zeta(c)}$ whenever $c \in W$. Hence the Mandelbrot set in the region W can be interpreted as

$$M \cap W = \{c \in W \mid c \in K_c\} = \left\{ c \in W \ \middle| \ \frac{1}{(Q_c^k)'(z)|_{z=\zeta(c)}} \varphi_c \circ Q_c^k(c) \in L(c) \right\} \ . \qquad (5)$$

To apply the above proposition, set $\Lambda = W$, $\lambda_0 = c_0$, $A(c) = L(c)$ and

$$u(c) = \frac{1}{(Q_c^k)'(z)|_{z=\zeta(c)}} \varphi_c \circ Q_c^k(c) \ .$$

We have proved in part a) that $c \mapsto L(c)$ satisfies the condition * at c_0, and $L(c)$ is $\rho(c)$-self-similar about 0. Moreover $c \mapsto \rho(c)$ is holomorphic, with $|\rho(c_0)| > 1$. It is clear that $u(c)$ is also holomorphic. As a consequence of the connectedness of M, Douady and Hubbard showed that $u'(c_0) \neq 0$. Set $L(c_0) = L$. Now since $M \cap W = M_u$, Proposition 4 shows that, for any $r > 0$,

$$D_H \left((\rho^n \tau_{-c_0} M)_r, (\frac{1}{u'(c_0)} L)_r \right) \to 0 \ .$$

Moreover, an elementary calculation (see [T1] for details) shows that the required μ is

$$\frac{1}{u'(c_0)} = \frac{(Q_{c_0}^k)'(z)|_{z=c_0}}{\dfrac{d}{dc}(Q_c^k(c))|_{c=c_0} - \dfrac{d}{dc}(\alpha(c))|_{c=c_0}} \ . \qquad (6)$$

This ends the proof of b).

Example. Let us take again the example of $c_0 = i$. To find the value of μ , we first obtain $(Q_{c_0}^k)'(z)|_{z=c_0} = (Q_i)'(z)|_{z=i} = 2i$,

$$\frac{d}{dc}(Q_c^k(c))|_{c=i} = \frac{d}{dc}(c^2 + c)|_{c=i} = (2c + 1)|_{c=i} = 2i + 1 \ .$$

The function $\alpha(c)$ is the solution near $i - 1$ of the implicit equation $Q_c^2(z) = z$, i.e. $(z^2 + c)^2 + c - z = 0$ for c close to i . Hence

$$\frac{d}{dc}(\alpha(c))|_{c=i} = -\frac{2((i-1)^2 + i) + 1}{2((i-1)^2 + i) \cdot 2(i-1) - 1} = \frac{2i - 1}{4i + 3} \quad \text{and} \quad \mu = \frac{2i}{2i + 1 - \dfrac{2i-1}{4i+3}} = 1 + \frac{1}{2} i \ .$$

Computer experiments confirm the similarity very impressively.

The proof of b') is also done in two steps:

Proposition 5 *Assume that we have a holomorphic motion $i : \Delta \times X \to \overline{\mathbb{C}}$ (where Δ denotes the unit disc) and an analytic mapping $v : \Delta \to \overline{\mathbb{C}}$, with $v(0) = z_0 \in X$, $v(\lambda) \not\equiv i(\lambda, z_0)$ (weak transversality). Set $M_v = \{\lambda \in \Delta \mid v(\lambda) \in X(\lambda)\}$. Then*

$$\text{H-dim}(M_v) \geq \lim_{r \to 0} \text{H-dim}(X \cap \Delta(z_0, r)) \ .$$

Proof. (sketch) For simplicity, we assume $v'(0) \neq 0$. In the simple case that $X(\lambda) \equiv X(0)$ for $\lambda \in \Delta$ we have $M_v = v^{-1}(X(0))$. Since v is bi-Lipschitz near 0, and hence preserves the Hausdorff dimension, we have $\text{H-dim}(M_v) \geq \text{H-dim}(X \cap \Delta(z_0, r))$ for some $r > 0$. As a consequence $\text{H-dim}(M_v) \geq \lim_{r\to 0} \text{H-dim}(X \cap \Delta(z_0, r))$.

Now let us come back to the setting of our proposition. We will apply Rouché's theorem to prove that, for $r \in]0, r_0[$ (where r_0 is a small constant) and $Y^r = v^{-1}(X \cap \Delta(z_0, r))$, there is $R_r > 1$ (with $R_r \to \infty$ as $r \to 0$) and a holomorphic motion: $j^r : \{|\mu| < R_r\} \times Y^r \to \overline{\mathbb{C}}$ such that $j^r(1, Y^r) \subset M_v$. Therefore

$$\text{H-dim}(M_v) \geq \text{H-dim}(j^r(1, Y^r)) \geq C(1/R_r)\text{H-dim}(Y^r) = C(1/R_r)\text{H-dim}(X \cap \Delta(z_0, r)) \ ,$$

where the existence of $C(1/R_r)$ in the second inequality is due to a non trivial property of holomorphic motions (see below). Furthermore $C(1/R_r) \to 1$ as $r \to 0$.

(More precisely by Slodkowski's theorem (see for example [D1]), the mapping $j^r(1, \cdot)$ extends to a $K(1/R_r)$-quasi-conformal mapping and $K(1/R_r) \to 1$ as $1/R_r \to 0$. On the other hand, by Mori's inequality K-quasi-conformal maps are $1/K$-bi-Hölder continuous. Furthermore a simple calculation shows that for any $1/K$-bi-Hölder continuous map j and any set Y, we have

$$(1/K) \cdot \text{H-dim}(Y) \leq \text{H-dim}(j(Y)) \leq K \cdot \text{H-dim}(Y) \ .$$

Setting $C(1/R_r) = 1/K(1/R_r)$, we get the desired inequality).

To define the constant R_r and the holomorphic motion j^r, we proceed as follows: Assume $z_0 = 0$ and $i(\lambda, 0) \equiv 0$. Since the family of holomorphic maps $\{i(\cdot, z)\}_{z\in X}$ is normal, and $i(\lambda, z) \neq 0$ for any $z \neq 0$ and any λ (by injectivity), any limit function of the family corresponding to a sequence $z_n \to 0$ must be the constant function 0. Fix $s < 1$ and $a > 0$ such that in $\Delta(0, s)$, $v(\lambda)$ is injective and $a|\lambda| \leq |v(\lambda)|$. Then for $b_r = \sup\{|i(\lambda, z)| \mid z \in X \cap \Delta(0, r),\ |\lambda| \leq s\}$, we have $b_r \to 0$ as $r \to 0$.

Fix r_0 such that $as > b_r$ for $0 < r < r_0$. Take $r \in]0, r_0[$, set $R_r = as/b_r$. Take $\mu \in \Delta(0, R_r)$ and $z \in X \cap \Delta(0, r)$. The equation $v(\lambda) - i_{\mu\lambda}(z) = 0$ has a unique solution $\lambda(r, \mu, z)$ in the disc $\Delta(0, \min\{s, s/|\mu|\})$ (we just apply Rouché's theorem here).

Now set $Y^r = v^{-1}(X \cap \Delta(0, r))$ and define $j^r : \{|\mu| < R_r\} \times Y^r \to \overline{\mathbb{C}}$ by $j^r(\mu, y) = \lambda(r, \mu, v(y))$. It is then easy to check that j^r is a holomorphic motion and $j^r(1, Y^r) \subset M_v$. ∎

Now we should adapt the above result to the situation of the boundary ∂M of the Mandelbrot set. It is much less trivial than the similarity case because we want M_v to be a subset of ∂M.

Lemma 6 *For the holomorphic motion in part a'), there are $z_0 \in X$, $c' \in \Delta' \subset \Delta$, with Δ' a neighborhood of c', and $v(c) = Q_c^N(0)$ for some $N > 0$ with $v(c') = i_{c'}(z_0)$, $v(c) \not\equiv i_c(z_0)$ such that $\{c \in \Delta' \mid v(c) \in X(c)\} \subset \partial M \cap \Delta'$, and $\lim_{r\to 0} \text{H-dim}\,(X_{c'} \cap \Delta(i_{c'}(z_0), r)) > 2 - \varepsilon$.*

Proof. (sketch). By compactness of X, there exists a point $z_0 \in X$ such that $\text{H-dim}(X) = \lim_{r\to 0} \text{H-dim}(X \cap \Delta(0, r)) > 2 - \varepsilon$. Let $i : \Delta(c_0, r_0) \times X \to \mathbb{C}$ be the holomorphic motion given in part a'). Since for c close to c_0 the mapping $i(c, \cdot)$ does not change too much the Hausdorff dimension (see the proof of the above proposition), there is a small neighborhood $\Delta' \subset \Delta(c_0, r_0)$ of c_0 such that $\lim_{r\to 0} \text{H-dim}\,(X(c) \cap \Delta(i(c, z_0), r)) > 2 - \varepsilon$ for $c \in \Delta'$.

Recall that F_n denotes the map $c \mapsto Q_c^n(0)$. The boundary of M coincides with the set

$$\{\ c \in \mathbb{C} \mid \text{ the family } \mathcal{F} = \{F_n\ ,\ n \in \mathbb{N}\ \} \text{ is not normal at } c\ \}$$

(see B. Branner's paper). By part a'), $c_0 \in \partial M$. We claim that there is $c' \in \Delta'$, an integer $N > 0$, such that $F_N(c') = i(c', z_0)$. For otherwise the family $\mathcal{F}$ would satisfy Montel's normality criterion at c_0 with respect to the two analytic functions $c \mapsto i(c, z_0)$ and one branch of $c \mapsto Q_c^{-1}(i(c, z_0))$, which contradicts the fact that $\mathcal{F}$ is not normal at c_0.

Set $v(c) = F_N(c)$. In order to apply the above proposition, we need to know that $v(c) \not\equiv i(c, z_0)$ and the set $M_v = \{c \in \Delta' \mid v(c) \in X(c)\}$ is a subset of ∂M.

One thing that was not explicitly stated in part a') is that, besides the other properties in a'), we have also $Q_{c_0}(X) = X$, moreover the mapping $i(c, \cdot)$ conjugates the dynamics, i.e.

$$i(c, Q_{c_0}(z)) = Q_c(i(c, z))\ . \tag{7}$$

As a consequence $Q_c(X(c)) = X(c)$ and $X(c) \subset J_c$.

There are (at least) two ways to see that $v(c) \not\equiv i(c, z_0)$: I. Since $c_0 \in \partial M$, there is $c'' \in \Delta' - M$. So $F_N(c'') = Q_{c''}^N(0) \notin J_{c''}$. But $i(c'', z_0) \in X(c'') \subset J_{c''}$. II. We may also use the normality argument. Assume $F_N(c) \equiv i(c, z_0)$ in Δ'. Then, for any integer $k > 0$,

$$F_{N+k}(c) = Q_c^k(F_N(c)) \equiv Q_c^k(i(c, z_0)) = i(c, Q_{c_0}^k(z_0))\ ,$$

where the last equality is due to the formula (7). So the family $\mathcal{F}$ is uniformly bounded in Δ', hence normal. This contradicts the fact that $\mathcal{F}$ is not normal at c_0.

To prove $M_v \subset \partial M$ we need Mañé-Sad-Sullivan's characterization of ∂M: We say that Q_c is J-stable at c_1 if there is a continuous map $h : \Delta(c_1, r) \times J_{c_1} \to \overline{\mathbb{C}}$ such that $h_c \equiv h(c, \cdot)$ is a conjugacy from (J_{c_1}, Q_{c_1}) to (J_c, Q_c) and $h_{c_1} = Id$. Then

$$\partial M = \{c_1 \in \mathbb{C} \mid Q_c \text{ is NOT } J\text{-stable at } c_1\ \}\ .$$

(This formula can be used to get another way to prove $v(c) \not\equiv i(c, z_0)$, for otherwise one can pull back the formula (7) to get a holomorphic motion of $\bigcup_n Q_{c_0}^{-n}(v(c_0))$. Since this is a dense subset of J_{c_0}, we can apply the λ-lemma of Mañé-Sad-Sullivan to show that Q_c is J-stable at c_0, thus a contradiction.)

Now assume that c_1 is a point in $M_v - \partial M$. So Q_c is J-stable at c_1, and admits a conjugacy map $h : \Delta(c_1, r) \times J_{c_1} \to \overline{\mathbb{C}}$. Decreasing r if necessary, we may assume $\Delta(c_1, r) \subset \Delta'$. Set $X_1 = i(c_1, X)$. We claim that $h|_{\Delta(c_1,r)\times X_1}$ must coincide with the holomorphic motion i. The reason is that both maps are continuous and $h(c, z) = i(c, z)$ for z any repelling periodic point, and repelling periodic points are dense in X_1 (this is because X_1 is a hyperbolic subset).

On the other hand, h must preserve the critical point, i.e. $h(c,0)=0$. So $h(c,v(c_1))=v(c)$. This gives rise to a contradiction since $v(c_1)=i(c_1,z_0)$ but $v(c)\not\equiv i(c,z_0)$.

Hence all the conditions required by the above proposition are satisfied. So

$$\text{H-dim}(\partial M)\geq \text{H-dim}(M_v)\geq \lim_{r\to 0}\text{H-dim}\left(X(c')\cap\Delta(i(c',z_0),r)\right)>2-\varepsilon\ .$$

This completes the proof of b'). ■

4. Appendix

1. An example of a self-similar set. Let $A\subset[0,1]$ denote the standard middle third Cantor set. Take all logarithmic spirals through points in A which in logarithmic coordinates are straight lines parallel to the vector $\log 3+2\pi i$. This set is ρ-self-similar for $\rho=3^t e^{2\pi i t}$ for any $t\in\mathbb{R}_+$.

2. Local connectivity. For c_0 a Misiurewicz point, [T2] constructed a sequence of Jordan curves Γ_n in the dynamical plane such that $A=\{c_0\}\cup\bigcup_n\Gamma_n$ is asymptotically ρ-self-similar and the set of bounded components U_n of $\overline{\mathbb{C}}-\Gamma_n$ forms a basis of nested neighborhoods of c_0 with $\overline{U}_n\cap K_{c_0}$ connected. Moreover there is a holomorphic motion $i:\Delta(c_0,r_0)\times A\to\overline{\mathbb{C}}$ with $A(c)$ asymptotically $\rho(c)$-self-similar. As $n\to\infty$, the sequence of subsets $\mathrm{P}\Gamma_n=\{c\in\Delta(c_0,r_0)\mid c\in\Gamma_n(c)\}$ bounds a nested sequence of neighborhoods W_n of c_0 in the parameter plane, with $\overline{W}_n\cap M$ connected. Applying Proposition 4 to $\mathrm{P}\Gamma_n$, we see that it is also asymptotically ρ-self-similar. In particular the diameters of W_n shrink to zero exponentially fast. This is a stronger statement than saying that M is locally connected at c_0.

3. Hausdorff dimension: Let (E,d) be a metric space. For $A\subset E$, denote by $|A|$ its diameter. For $X\subset E$, $t>0$, $\varepsilon>0$, we define

$$m_t^\varepsilon(X)=\inf_{\{A_j\}}\left\{\sum_{j\in\mathbb{N}}|A_j|^t\ \middle|\ 0<|A_j|\leq\varepsilon,\ X\subset\bigcup_{j\in\mathbb{N}}A_j\right\}\ .$$

Fixing t, $m_t^\varepsilon(X)$ increases as ε decreases. We can then define $m_t(X)=\lim_{\varepsilon\to 0}m_t^\varepsilon(X)=\sup_{\varepsilon>0}m_t^\varepsilon(X)$. Note that $m_t(X)$ can be ∞. An easy calculation shows that if for some t, $m_t(X)<\infty$, then $m_T(X)=0$ for any $T>t$. As a consequence, there is a unique number $\delta\in[0,\infty]$ such that $m_t(X)=\infty$ for $t<\delta$ and $m_t(X)=0$ for $t>\delta$. This δ is called the Hausdorff dimension of X.

It is not easy to calculate the Hausdorff dimension in general. However, in the following situation there is an easy lower bound: Let $G:V\to U$ be an analytic covering, with U an open disc, V a finite union of open discs with disjoint closures and $\overline{V}\subset U$. Then for the non-escaping set $X=\{z\in V\mid G^m(z)\in V \text{ for all } m>0\}$, one has:

$$\text{H-dim}(X)\geq\frac{\log(\text{number of components of } V)}{\log(\max|G'(z)|)}\ .$$

The set X is a special example of *hyperbolic sets* and is automatically stable under perturbation.

4. Proof of theorem 1.2.a'). As an example, we give the main steps to show that there are $c_n\to c=1/4$, $c_n\in\partial M$ and hyperbolic subsets $X_n\subset\partial K_{c_n}$ such that $\text{H-dim}(X_n)\to 2$ as $n\to\infty$. The technique is called geometric limits of a parabolic map, and parabolic implosion.

A similar study can be done for each c in a dense subset of ∂M (namely the set of roots of primitive hyperbolic components).

Denote by f the map $z \mapsto z^2 + 1/4$. There will be two holomorphic functions g and h (the first and a second geometric limit of f) generating the family of maps

$$\mathcal{L} = \{f^k g^l h^m \mid k, l, m \in \mathbb{Z}\ ,\ m > 0\}$$

(with a certain convention on f^{-1} and g^{-1}) satisfying the following two properties:

4.1. There exists a sequence $c_n \to 1/4$ (with $c_n \in \partial M$, but one can also choose c_n in the main cardioid or $c_n \in \mathbb{C} - M$) such that for each $G_i \in \mathcal{L}$, there are integers $j(n,i) \to +\infty$ such that $Q_{c_n}^{j(n,i)}$ converges to G_i uniformly on compact sets in the domain of definition of G_i.

4.2. There exists a small neighborhood U of $1/2$ and constants $a, C, C' > 0$ such that for large $\eta > 0$, there are open sets $U_1, \cdots, U_N$, with $N > a\eta^2$, $\overline{U}_i \cap \overline{U}_j = \emptyset$, $\overline{U_i} \subset U$, and $G_i \in \mathcal{L}$ such that $G_i|_{\overline{U}_i} : \overline{U}_i \to \overline{U}$ is bijective and $C\eta(\log\eta)^2 < |G_i'|_{\overline{U}_i}| < C'\eta(\log\eta)^2$.

As a consequnce, for n large, $i = 1, \cdots, N$, there is $j(n,i)$ large and $U(n,i)$ close to U_i such that $Q_{c_n}^{j(n,i)}$ maps $U(n,i)$ bijectively onto U with derivative close that of G_i. For $X(n)$ the non-escaping set, the formula in Appendix 3 gives us:

$$\text{H-dim}(X(n)) \geq \frac{2\log\eta + \log a}{\log\eta + 2\log\log\eta + \text{constant}} > 2 - \varepsilon\ .$$

We will skip the proof of 4.1 (which can be found in the papers of Shishikura and in [D2]) and give a sketch of the construction of g, h, U_i, U, G_i and the estimate of $|G_i'|$.

a) Denote by B the basin of the parabolic point $1/2$ for f. By classical results there are holomorphic surjective maps $\varphi_1 : \mathbb{C} \to \mathbb{C}$, $w \mapsto z$ and $\Phi_1 : B \to \mathbb{C}$, $z \mapsto w$ such that $f \circ \varphi_1 = \varphi_1 \circ T$ and $\Phi_1 \circ f = T \circ \Phi_1$, where T denotes the translation $w \mapsto w + 1$ (the mappings φ_1, Φ_1 are called *Fatou coordinate changes* of f). Set $g = g_1 = \varphi_1 \circ \Phi_1 : B \to \mathbb{C}$; $\tilde{g}_1 = \Phi_1 \circ \varphi_1 : \varphi_1^{-1} B \to \mathbb{C}$ and $\overline{g}_1 = \pi \circ \tilde{g}_1 \circ \pi^{-1} : \pi(\varphi_1^{-1}(B)) \to \mathbb{C}^*$ (where $\pi(w) = e^{2\pi i w}$). Then there exist choices (unique up to addition of integers) of φ_1, Φ_1 such that $\tilde{g}_1(w) = w + o(1)$ as $Im(w) \to +\infty$ and $\overline{g}_1(0) = 0$, $\overline{g}_1'(0) = \overline{g}_1''(0) = 1$. For this $\overline{g}_1$, we have $\overline{g}_1(\infty) = \infty$ and $|\overline{g}_1'(\infty)| > 1$.

b) Because B is simply connected and contains only one critical value, the immediate basin B' of 0 for the map $\overline{g}_1$ has the same property. Similarly one can find φ_2, Φ_2 (but with $\varphi_2(\mathbb{C}) = \mathbb{C}^*$ instead) and define $g_2, \tilde{g}_2, \overline{g}_2$ with the same asymptotic behavior at 0 and ∞. There are lifts of g_2 by π^{-1} and $\varphi_1 \circ \pi^{-1}$ successively. We call them $\tilde{h}$ and h.

c) Here is the diagram of our construction:

$$\begin{array}{ccccccc}
U_i & \xleftarrow{\varphi_1} & & \xleftarrow{\pi_0^{-1}} & & \xleftarrow{\varphi_2} & W_i \\
G_i \downarrow & & \tilde{s} \downarrow & & s \downarrow & & \downarrow T^{l_i} \tilde{g}_2^{m_i} \\
U & \xleftarrow{\varphi_1} & & \xleftarrow{\pi^{-1}} & & \xleftarrow{\varphi_2} & W
\end{array}$$

where $\tilde{s} = T^k \tilde{g}_1^{l_i} \tilde{h}^{m_i}$ and $s = \overline{g}_1^{l_i} g_2^{m_i}$. The other terms are going to be defined below.

Fix U a small disc neighborhood of $1/2$. There is a disc W and a choice of π^{-1} such that $\varphi_1 \circ \pi^{-1} \circ \varphi_2$ maps $\overline{W}$ onto $\overline{U}$ bijectively with bounded derivative.

Since ∞ is repelling for $\overline{g}_2$, the map $\tilde{g}_2$ behaves like a translation $w \mapsto w + \mu$ with $Im(\mu) > 0$ as $Im(w) \to -\infty$. So for $m \in \mathbb{N}$ there are W_m' disjoint discs such that $\tilde{g}_2 W_m' = W_{m-1}'$,

$Im(W'_m) \to -\infty$ as $m \to \infty$ and $\tilde{g}_2^m : \overline{W'}_m \to \overline{W}$ is a bijection with bounded derivative (independent of m).

Fix $\eta > 0$ large. In the rectangle $R(\eta) = \{w,\ |\Re w| < \eta\ ,\ Im(w) \in [-\eta, -2\eta]\}$, there are N disjoint discs $W_1, \cdots, W_N$, with $N > a\eta^2$, and $l_i, m_i \in \mathbb{Z}$, $m_i > 0$ such that $T^{l_i}\overline{W}_i = \overline{W'}_{m_i}$. Therefore for each i the map $T^{l_i}\tilde{g}_2^{m_i} : \overline{W}_i \to \overline{W}$ is a bijection with bounded derivative (independent of i and η).

Let $\pi_0^{-1} : \mathbb{C}^* - \mathbb{R}^+ \to \{w | 0 < \Re w < 1\}$ be the special branch of π^{-1}. Set $U_i = \varphi_1 \circ \pi_0^{-1} \circ \varphi_2(W_i)$.

One can easily check that for η large and a good choice of $\tilde{h}$, we have $\overline{U}_i \subset U$ and there is $k \in \mathbb{Z}$ (independent of i and η) such that $f^k g^{l_i} h^{m_i} \overline{U}_i = \overline{U}$. Set $G_i = f^k g^{l_i} h^{m_i}|_{U_i}$.

d) The derivative $|G'_i|$ is controlled by $1/|(\varphi_1 \circ \pi_0^{-1} \circ \varphi_2)'(w)|$, $w \in \overline{W}_i \subset R(\eta)$, the rest has bounded effects. On the other hand, when η is large, both φ_1 and φ_2 can be approximated by $I(w) = -1/w$. A simple calculation shows that $|G'_i| \sim \eta(\log \eta)^2$.

Some key words: dynamical spaces, parameter space, product space, implicit function theorem, Rouché's theorem, holomorphic motion.

References

[D1] A. Douady, Prolongement de mouvements holomorphes [d'après Slodkowski et autres], Séminaire Bourbaki, number 755, November 1993.

[D2] A. Douady, Does a Julia set depend continuously on the polynomial? Proceedings of Symposia in Applied Mathematics, Vol. **49**, 1994

[S1] M. Shishikura, The parabolic bifurcation of rational maps, 19° Colóquio Brasileiro de Matemática, IMPA 1993.

[S2] M. Shishikura, The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets, preprint SUNY Stony Brook, 1991/7.

[S3] M. Shishikura, The boundary of the Mandelbrot set has Hausdorff dimension two, S.M.F. Astérisque 222, 1994

[T1] Tan Lei, Similarity between Mandelbrot set and Julia sets, Commun. Math. Phys. 134, 587-617, 1990.

[T2] Tan Lei, Voisinages connexes des points de Misiurewicz, Ann. Inst. Fourier, Grenoble, 42, 4, 707-735, 1992.

UMR 128 CNRS, ENS Lyon, 46 allée d'Italie, 69364 Lyon 07, France.
Department of Mathematics, University of Warwick, Coventry CV4 7AL, U.K.

Classification in Algebraic Geometry

A Short Course organized by Rosa M. Miró–Roig and Raquel Mallavibarrena

The session consisted of three talks organized with the aim of present to a non-specialized audience some basic facts and main problems that belong to this relatively recent but vast field of Algebraic Geometry.

The first talk was given by Margarida Mendes–Lopes and it was essentially an introduction, with basic concepts and definitions, all this with the goal of making the other two talks understandable.

Mireille Martin–Deschamps spoke in the second place. She presented some aspects of the problem of classification of space curves: results, techniques,...

The third talk was given by Emilia Mezzetti. She also focused on a classification problem, the one for projective varieties of small codimension. Here, as well as in the previous talk, the use of sheaves, schemes and cohomology was proven to be a powerful tool. In fact, these techniques have made possible to talk of the so–called modern algebraic geometry.

There was also a kind of example session where several computations were made concerning some well - known problems that have an elementary treatment.

Rosa Maria Miró–Roig

Proc. of the 7^{th} EWM meeting, Madrid, 1995

On some notions in algebraic geometry

Margarida Mendes Lopes

Universidad de Lisboa, Portugal
mmlopes@ptmat.lmc.fc.ul.pt

Foreword. This paper is an expanded version of the talk given to the 7^{th} meeting of EWM. It was meant as a (very basic) introductory talk and does not propose to be a survey on surface theory.

1. Some notions

We will be dealing with the projective space $\mathbb{P}^n$ over the complex numbers $\mathbb{C}$. The projective space $\mathbb{P}^n$ is the set of 1-dimensional vector spaces of $\mathbb{C}^{n+1}$ and for each point $p \in \mathbb{P}^n$ its homogeneous coordinates $(x_0, ..., x_n)$ are defined up to scalar multiple.

Given an homogeneous polynomial $f(x_0, ..., x_n)$ its locus of zeroes is a well-defined subset of $\mathbb{P}^n$, called an algebraic hypersurface.

More generally a *projective closed algebraic set* $V \subset \mathbb{P}^n$ is the common locus of zeroes of a finite set of homogeneous polynomials, i.e. $V = \{x = (x_0, ..., x_n) : f_1(x) = ...f_k(x) = 0\}$. A closed algebraic set is a *projective algebraic variety* if V is irreducible (i.e. V cannot be written as the union of two proper subsets which are algebraic sets).

Given an algebraic closed set we consider the ideal $I(V)$ formed by all homogeneous polynomials which vanish identically on the points of V. This ideal is a finitely generated homogeneous ideal and it will be a prime ideal if and only if V is irreducible.

Associated to a variety $V \subset \mathbb{P}^n$ we have its *dimension* and its *degree*.

There are various equivalent ways of defining the dimension, but possibly, since we are talking about algebraic varieties over the complex numbers, the easiest way is defining the dimension of V as its dimension as a complex topological space. Other possible ways (which are equivalent) are defining the dimension as $d := n - r$, where r is the maximum rank of the matrix $(\frac{\partial g_i}{\partial x_j})$ evaluated over every point $p \in V$, where g_i are a set of generators for $I(V)$ or equivalently we can define the dimension as the maximal m such that V projects surjectively to $\mathbb{P}^m$.

Now the degree can be again defined alternatively as the number of points of intersection of V with a general linear subspace of complementar dimension, or then as the number of points on a general fibre of a projection which realizes the dimension. For an hypersurface the degree is simply the degree of a polynomial generating $I(V)$ and the dimension is $n - 1$.

A variety of dimension 1 is a *curve*, whilst one of dimension 2 is a *surface*. As simple examples one has for instance the cuspidal cubic curve in $\mathbb{P}^2$ defined by $y^2z - x^3 = 0$, the twisted cubic curve in $\mathbb{P}^3$ which is defined by the equations

$$x_0x_2 - x_1^2 = 0, \quad x_0x_3 - x_1x_2 = 0 \text{ and } x_1x_3 - x_2^2 = 0,$$

or a complete intersection of two quadrics in $\mathbb{P}^4$.

Let us point out that similarly to what happens with manifolds we can also give a notion of an abstract algebraic projective variety which does not depend on the ambient space we are considering and the dimension does not depend on the projective space we are considering. Of course the degree will not be invariant in this context. Anyway for our purposes it will be enough to keep this definition as the locus of solutions of a finite system of polynomial equations in mind.

Given an algebraic variety we can define a *rational function* on it as being a function defined locally by a quotient of homogeneous polynomials of same degree. A *rational map* between algebraic varieties $V \to W \subset \mathbb{P}^n$ is defined by $v \to (1, f_1(v), ..., f_n(v))$ where f_i is a rational function on V. Now the main thing to remark is that a rational map is not necessarily defined everywhere because there are points which are poles of every rational function appearing. A morphism is a rational map everywhere defined and we will say that a rational map is birational if it has an inverse which is a rational map. In this case basically what happens is that the two varieties are the "same" in the complement of a closed algebraic set.

We will be mainly concerned with *smooth* (also called non-singular) projective varieties which are those such that the rank of the matrix above is constant at every point. Although apparently we are missing out a lot that is not the case (at least up to birational equivalence) due to Hironaka's desingularization theorem, which says it is possible given a variety to find a non-singular variety which is birational to it (by blowing-ups).

2. What is classification?

There are various problems of classification that arise. One is exactly classifying algebraic projective varieties up to birational equivalence and finding a way of describing all equivalence classes and this puts us into theory of moduli.

Another is trying to find out which smooth varieties and of a given dimension live in some $\mathbb{P}^n$, and what degrees can turn up. Yet another is trying to classify according to the the "minimal" number of generators for its homogeneous ideal in $\mathbb{P}^n$.

It turns out that every smooth algebraic variety of dimension d can be isomorphically projected in $\mathbb{P}^{2d+1}$. So for instance one possible problem of classification is trying to find for given degrees which smooth varieties of dimension d are embedded (and with which degrees) in $\mathbb{P}^k$ where $k \leq 2d$.

Suppose then we want to classify smooth varieties of a given dimension up to birational equivalence. We would like to have a way of individuating in each birational equivalence class a representative. For curves it can be shown that each birational equivalence class contains a unique smooth curve. In the case of surfaces this is no longer true, but nevertheless we can find minimal smooth surfaces. A surface S is said to be *minimal* if any birational morphism of $S \to S'$, with S' another smooth surface, is an isomorphism. Given any smooth surface X it is always possible (by blowing down the exceptional curves) to find a smooth minimal surface S together with a birational morphism $X \to S$. S is said to be a minimal model of X. Now what happens for surfaces is that except for the surfaces in a certain class (i.e.

those with $kod(X) = -\infty$, see section 3) every birational equivalence class contains an unique minimal non-singular model and this helps a lot with the classification.

In this talk I'll focus precisely on the classification of surfaces up to birational equivalence, more precisely on the usually called Enriques-Kodaira classification. For this I will need :

3. Some more notions

A *divisor* D on a smooth variety X is a formal finite sum $D = \sum a_i C_i$ where a_i is an integer and C_i is any subvariety of X of codimension 1. A divisor is *effective* if $a_i \geq 0$ for every i. The set of all divisors with the obvious operation is a group. To every non-zero rational function f on X corresponds the divisor of f, (f) which is defined as being the difference $(f)_0 - (f)_\infty$ of the two effective divisors $(f)_0$, $(f)_\infty$ given respectively by the zeroes and the poles of f. Two divisors are linearly equivalent if their diference is the divisor of some rational function.

If X is a smooth surface we can define the *intersection number* of two divisors. This intersection number for two curves meeting tranversally at smooth points is exactly the number of points in which the curves meet.

In the complex case it can be defined as an intersection number in homology. In fact each divisor D corresponds to a homology class in $H_2(X, \mathbb{Z})$. Given two divisors D_1, D_2 we can define the intersection number $D_1 \cdot D_2$ as being the intersection number of their homology classes. This intersection number can also be defined in purely algebraic terms.

Given a divisor D on an algebraic variety one can associate to it a vector space $\mathcal{L}(D)$ consisting of 0 and the rational functions f such that $div(f) + D$ is an effective divisor. If this vector space is non-zero and $n+1$ dimensional we can consider the map $\phi_D : X \to \mathbb{P}^n$ given by the functions of a basis of $\mathcal{L}(D)$.

In particular one can speak about the pluricanonical maps of the variety X which are associated to the multiples of the canonical divisors of X. What are those?

The canonical divisors can be defined in purely algebraic terms but in this context, since we are considering varieties over the complex numbers, we are going to use the language of complex manifolds.

In fact any smooth algebraic variety over $\mathbb{C}$ is a compact complex manifold with the holomorphic structure inherited from $\mathbb{P}^n$.

Let us remark that the class of compact complex manifolds is bigger than the class of smooth algebraic varieties. A compact complex manifold which is also an algebraic variety is said to be (projective) algebraic. Nevertheless one has: :

Theorem 1 *Every compact complex manifold of dimension 1 (Riemann surface) is algebraic. A compact complex manifold X of dimension 2 is algebraic if and only if $a :=tr.deg_{\mathbb{C}}\ \mathcal{M}(X) = 2$, where $\mathcal{M}(X)$ is the field of meromorphic functions on X.*

If the compact complex manifold X is projective algebraic it turns out that any meromorphic function is actually a rational function.

Associated with each compact complex manifold of dimension n one has the *canonical line bundle* which is $\mathcal{K}_X = \wedge^n \mathcal{T}_X^\vee$, where $\mathcal{T}_X^\vee$ is the (holomorphic) cotangent bundle. So $\mathcal{K}_X$ is the line bundle which holomorphic sections are the n-forms on X,(for instance in the surface case locally if z_1, z_2 are local coordinates local holomorphic sections are given by $f(z_1, z_2) dz_1 \wedge dz_2$ where $f(z_1, z_2)$ is an holomorphic function).

Having the canonical bundle we define the pluricanonical bundles $\mathcal{K}_X^{\otimes m}$.

The *canonical (resp. pluricanonical) divisors* on the smooth projective variety X are then defined as the divisors obtained as the divisor of zeroes minus the divisor of poles of a meromorphic section of $\mathcal{K}_X$ (resp. $\mathcal{K}_X^{\otimes m}$). The canonical divisors are denoted by K_X and the pluricanonical by mK_X (omiting the subscript when there is no danger of confusion).

This takes us to yet another definition: *Kodaira dimension of* X, $kod(X)$, as the maximum dimension of the image of ϕ_{mK}, for $m \in \mathbb{N}$. One has always $kod(X) \leq \dim\ (X)$, and if $kod(X) = \dim\ X$, X is said to be of *general type.* If $\mathcal{L}(mK) = 0$, by convention dim $\phi_{mK}(X) = -\infty$.

The Kodaira dimension is a birational invariant and this gives us a very rough classification of varieties with respect to birational equivalence which is done by dividing them in various classes according to the Kodaira dimension.

The Kodaira dimension in fact can be defined for any compact complex manifold. In the case of surfaces it turns out that if the compact complex surface X has $kod(X) = 2$, then it is an algebraic surface.

4. The rough classification of minimal surfaces

For surfaces one can describe the surfaces in some of these classes in more detail. To explain this still rough classification of we will need some more notions.

A surface S is called *elliptic* if it admits an elliptic fibration, i.e. a morphism f onto a smooth curve B such that almost all fibres $f^{-1}(p)$ are smooth elliptic curves (i.e. Riemann surfaces of genus 1).

Associated with a surface we have various invariants. Some are linked to the canonical divisors :

$p_g := dim_{\mathbb{C}}\mathcal{L}(K)$ =number of linearly independent 2-forms (geometric genus);

$p_i := dim_{\mathbb{C}}\mathcal{L}(mK)$ (plurigenera)

$q :=$ number of linearly independent 1-forms, called the irregularity;

$\chi(\mathcal{O}_S) := p_g - q + 1$;

K^2 (also denoted by c_1^2):= the self intersection of a canonical divisor;

c_2:= topological Euler-Poincaré characteristic.

These invariants are linked by various relations of which maybe the most relevant is the *Noether's formula:*

$$K_S^2 + c_2 = 12\chi(\mathcal{O}_S).$$

Let us notice that all these invariants can be recovered from the topology of S. Also p_g, p_i and q (and $\chi(\mathcal{O}_S)$) are birational invariants whilst K^2 and c_2 are not. For birational equivalence classes $\mathcal{C}$ with a unique minimal model S one has that K_S^2 is maximum for the surfaces in $\mathcal{C}$.

Let us notice also that in spite of not looking so via Hodge theory it can be shown that all these invariants are in fact topological and computable from triangulations.

Now we can give:

The rough classification of minimal algebraic surfaces (Enriques):

(I) *Kodaira dim* $= -\infty$: These are the surfaces for which $p_i = 0$ for all $i \in \mathbb{N}$ and are either $\mathbb{P}^2$ or ruled surfaces (i.e. a $\mathbb{P}^1-$bundle over a smooth curve) .

(II) *Kodaira dimension* $= 0$: These are the surfaces for which $p_i \leq 1$ for all $i \in \mathbb{N}$ and $p_i \neq 0$ for some $i \in \mathbb{N}$ and are divided in the following different classes:

- **(a)** bielliptic (sometimes called hyperelliptic) surfaces - admit a locally trivial fibration over an elliptic curve with elliptic fibre ($p_g = 0$, $q = 1$)
- **(b)** K3 surfaces: These are the simply connected surfaces with $K_X = \mathcal{O}_X$ ($p_g = 1$, $q = 0$).
- **(c)** Enriques surfaces: These are such that $K_X \neq \mathcal{O}_X$ and $2K_X = \mathcal{O}_X$ ($p_g = 0$, $q = 0$) and are quotients of K3 surfaces by a fixed point free involution.
- **(d)** Abelian surfaces: These are quotients of $\mathbb{C}^2$ by some lattice ($p_g = 1$, $q = 2$).

(III) *Kodaira dimension* 1: These surfaces have $p_i \geq 1$ for some $i \in \mathbb{N}$ but the corresponding pluricanonical image is a curve and are all elliptic surfaces.

(IV) *Kodaira dimension* 2: These are the surfaces for which some pluricanonical image is a surface and are called *surfaces of general type.*

Examples of surfaces in **(I)** are the projective plane, any smooth surface of degree $n-1$ in $\mathbb{P}^n$, any surface isomorphic to the product $\mathbb{P}^1 \times C$, with C a smooth curve.

Let us notice that the surfaces with Kodaira dimension 2 form a much "bigger" class. For instance for the smooth surfaces in $\mathbb{P}^3$ which are hypersurfaces, hence defined by some polynomial of degree d, one has that if $d = 2, 3$, S is birationally equivalent to $\mathbb{P}^2$, if $d = 4$ S is a K3-surface, whilst if $d \geq 5$ S is of general type.

Let us also remark that this classification can be extended to compact complex surfaces, not necessarily algebraic. For details see [BPV] or [s-Pe].

5. Surfaces of general type, geographical questions and pluricanonical models

"Most" surfaces will be of general type and one does not have a neat description like the ones for Kod≤ 1. One has:

Theorem 2 (Bombieri [B]) *Let S be a minimal smooth surface of general type. Then for $m >> 0$, ϕ_{mK} is a birational morphism and the images $\phi_{mK}(S)$ are all isomorphic. Furthermore $\phi_{mK}(S)$ is a surface having at most double points as singularities, and such that its minimal desingularization is isomorphic to S.*

Corollary 3 *Minimal surfaces of general type with given with given K^2, c_2 (or equivalently given (K^2, $\chi(\mathcal{O}_S)$) are surfaces of given degree in a given $\mathbb{P}^N$ and therefore are parametrized by a finite union of irreducible algebraic varieties (i.e belong to a finite number of families).*

By the Riemann-Roch theorem K^2, $\chi(\mathcal{O}_S)$ completely determine p_i, $i > 1$ for minimal surfaces of general type and hence we have this corollary, which Gieseker used to prove the existence of a coarse moduli space for surfaces of general type:

Theorem 4 (Gieseker) *For each pair of integers such that $9x \geq y \geq 2x - 6$ there exists a coarse moduli space $\mathcal{M}_{x,y}$ for all the isomorphism classes of minimal surfaces of general type with invariants $K^2 = y$ and $\chi(\mathcal{O}_S) = x$. Furthermore $\mathcal{M}_{x,y}$ is the union of a finite number of quasi-projective varieties (in particular it has a finite number of connected components).*

Where do these relations appearing in Gieseker's theorem come from? For minimal surfaces of general type one has the following relations between the invariants:

(i) $K^2 > 0$, $c_2 > 0$ (and $\chi(\mathcal{O}_S) > 0$).

(ii) $K^2 + c_2 = 12\chi(\mathcal{O}_S)$ (Noether's formula again).

(iii) (M.Noether-Bombieri) $K^2 \geq 2\chi(\mathcal{O}_S) - 6$ and if $q > 0$, $K^2 \geq 2\chi(\mathcal{O}_S)$.

(iv) (Bogomolov-Miyaoka-Yau) $K^2 \leq 3c_2$ (equivalently $K^2 \leq 9\chi(\mathcal{O}_S)$).

Within these restrictions one type of question (known usually as a geographical question) is for what pairs of natural numbers (n, m) satisfying the above relations there are minimal surfaces of general type with $K^2 = n$, $\chi(\mathcal{O}_S) = m$.

For what concerns geographical questions roughly the answer is for almost all pairs and this is proved by constructing very special surfaces.

The type of geographical results one has:

Theorem 5 (chronologically Persson [P], Sommese [S], Chen [C1], [C2], Ashikaga [A]) *For each pair of natural numbers (x, y) such that $9x - 347 \geq y \geq 2x - 6$ or $8x \geq y \geq 2x - 6$ (with the possible exceptions of $y = 8x - d, d = 1, 2, 3, 5, 7$) there exists a minimal surface S of general type such that $K_S^2 = x$, $\chi(\mathcal{O}_S) = y$.*

Another type of questions are the so called botanical questions: given a pair of numbers $(K^2, \chi(\mathcal{O}_S))$ (or equivalently (K^2, c_2) in the allowed region classify all minimal surfaces of general type with these invariants, describe families and if possible describe the moduli space .This last is almost impossible as follows from the following

Theorem 6 (Catanese [Ca1], [Ca2], [Ca3], Manetti [M]) *For all $n \in \mathbb{N}$ there is a pair $(K^2, \chi(\mathcal{O}_S))$ such that the moduli space has more than n irreducible components, pairwise of different dimensions.*

Nevertheless in some cases one has descriptions of the moduli spaces, like the ones given by Horikawa for surfaces with $K^2 - 2\chi(\mathcal{O}_S) - 6 \leq 2$ (see [Ho]).

There are a lot of other problems on surfaces I did not refer to, as the problems of equivalence of differentiable stuctures on the topological manifold underlying an algebraic surface or which of the above properties and classification hold for surfaces defined over a field of characteristic p.

References

This set of references is very incomplete. For an extended bibliography on surfaces see [sC].

Basic books:

[GH] P.Griffiths, J.Harris *Principles of algebraic geometry*, J.Wiley and Sons, 1978

[Ha] R.Hartshorne, *Algebraic geometry*, Springer, 1977

[H] J.Harris, *Algebraic geometry: a first course*, GTM 133, Springer, 1992

[R] M.Reid, *Undergraduate algebraic geometry*, LMS Student Texts **12**, C.U.P.

[S] I.R.Shafarevich *Basic algebraic geometry*, Springer

Basic theory of surfaces:

[Be] A.Beauville *Surfaces algébriques complexes*, Astérisque 126 (1985), Paris

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Ergebnisse der Math. (3) 4, Springer-Verlag, 1984

Surveys:

[sC] F.Catanese *(Some) old and new results on algebraic surfaces*, Proceedings of the First European Congress of Mathematicians (1992)

[sP] U.Persson *An introduction to the geography of surfaces of general type*, Proc. Symp. in Pure Math., 46 (1987), 195-218

[sPe] C.Peters *Introduction to the theory of compact complex surfaces*, Report of the Univ. of Leiden, October 1990

Papers refered to:

[A] T.Ashikaga, *A remark on the geography of surfaces with birational canonical morphisms*, Math. Ann, 290 (19910, 63-76

[B] E.Bombieri, *Canonical models of surfaces of general type*, Publ. Math. IHES, 42 (1973), 171-219

[Ca1] F.Catanese, *On the moduli space of surfaces of general type*, J. Diff. Geom., 19 (1984), 483-515

[Ca2] F.Catanese, *Connected components of moduli spaces*, J. Diff. Geom., 24 (1986), 395-399

[Ca3] F.Catanese, *Automorphisms of rational double points and moduli spaces of surfaces of general type*, Comp. Math., 61 (1987), 81-102

[C1] Z.Chen, *On the geography of surfaces - simply connected minimal surfaces with positive index*, Math. Ann., 277 (1987), 141-164

[C2] Z.Chen, *On the existence of algebraic surfaces with preassigned Chern numbers*, Math.Z., 206 (1991), 241-254

[Ho] E.Horikawa *Algebraic surfaces with small* c_1^2, I, Ann. Math., 104 (1976, 357-387. II, Invent. Math., 37 (1976), 121-155. III, Invent. Math., 447 (1978), 209-248. IV, Invent. Math., 50 (1979), 103-128.

[M] M.Manetti, *On some components of moduli spaces of surfaces of general type*, Publ. 137 Scuola Normale Pisa (1992)

[P] U.Persson, *On Chern invariants of surfaces of general type*, Comp. Math., 43 (1981), 3-58

[S] A.J.Sommese, *On the density of ratios of Chern numbers of algebraic surfaces*, Math.Ann., 268 (1984), 207-221

Proc. of the 7^{th} EWM meeting, Madrid, 1995

On classification of algebraic space curves, liaison and modules

Mireille Martin-Deschamps

École Normale Supérieure, Paris, France
deschamps@dmi.ens.fr

1. Smooth space curves

In the first paper on algebraic geometry, Margarida Mendes Lopes explained the structure of the projective space over the field $\mathbb{C}$. A *space curve* is a closed algebraic subset C of the projective 3-space $\mathbb{P}^3$, which has dimension 1 over $\mathbb{C}$ (note that it has dimension 2 over $\mathbb{R}$). It is defined by homogenous polynomials $(F_1, \ldots, F_r)$ belonging to the ring of polynomials in 4 variables $S = \mathbf{C}[X_0, X_1, X_2, X_3]$. Since a single polynomial defines a surface (of dimension 2 over $\mathbb{C}$), we have $r \geq 2$.

The projective 3-space $\mathbb{P}^3$ is obtained by gluing together 4 affine spaces, so a space curve C can be covered by 4 affine pieces. If we want to study *local* properties of the curves, that is properties in a neighbourood of a point, we can always restrict ourself to a point contained in an affine piece of the curve. For example, we say that a point $P = (a_0, a_1, a_2, a_3)$ of C is *smooth* (or that C is smooth at P) if the tangent space to C at P has dimension 1. To express it algebraically, we can suppose that $a_0 = 1$ and work in the open subset $X_0 \neq 0$. Let $f_i(x_1, x_2, x_3) = F_i(1, x_1, x_2, x_3)$, for $i = 1, \ldots, r$. Then P is a smooth point of C if and only if the rank of the matrix $(\frac{\partial f_i}{\partial x_j}(P))$ is 2.

A curve is *smooth* if and only if it is smooth at every point.

Examples : the curve defined by the polynomials (X_1, X_2) is smooth (it is a line), and the curve defined by the polynomials $(X_0X_1^2 - X_2^3, X_3)$ is not smooth (it has a cusp at the point $(1, 0, 0, 0)$).

It is possible to show that a "general" plane intersects transversally the curve in d distinct points, where the number d, which doesn't depend on the plane, is called the *degree* of the curve (a plane which is not general is a plane which is tangent).

Now a question : why are we interested in space curves ? We can of course define projective curves in $\mathbb{P}^n$ for $n > 3$, but one proves that, if C is a smooth curve contained in $\mathbb{P}^n$ for $n > 3$, there exists a projection π from $\mathbb{P}^n$ to an hyperplane $\mathbb{P}^{n-1}$ such that the restriction of π to C is an algebraic isomorphism from C onto its image. In other words, every smooth projective curve can be embedded in $\mathbf{P}^3$. On the other hand plane curves are in some sense special.

Example : let J be the ideal generated by the 2-minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \end{pmatrix}$$

The 2 quadric surfaces defined respectively by $X_0X_2 - X_1^2$ and $X_1X_3 - X_2^2$ have a line (defined by (X_1, X_2)) in common. Their intersection is a curve of degree 4 (by Bezout theorem), therefore it is the union of the line and a curve C of degree 3, called a *twisted cubic*, whose ideal of zeroes is J. More generally, any curve projectively equivalent to C is called a twisted cubic. If one projects C to a plane, one gets a plane cubic curve with a singularity.

2. Complete intersections. Liaison.

We have seen that a space curve is defined by at least two homogenous polynomials. From this point of view, the simplest case is therefore the case when the curve is defined by exactly two homogenous polynomials F and G (without common factor, since the dimension is 1). Such a curve is called a *complete intersection* (of two surfaces) (c.i. for short). Its degree is simply the product of the degrees of F and G. For example, plane curves of degree d are c.i. of a plane and a surface of degree d.

Like plane curves, c.i. are in some sense special. The smooth curves of least degree which are not c.i. are the twisted cubics. In fact, if the degree of a c.i. is a prime number, one of the two surfaces has degree 1, so is a plane. And we have seen that the twisted cubics are not plane.

Let again C be a twisted cubic. We have seen that it is contained in the intersection of two quadrics (which is a c.i.), and that the "residual" intersection is a line. We say that the cubic and the line are *linked* by the c.i.

Definition Two curves C and C' without common component are linked (by a complete intersection X) if X is the union of C and C'.

In fact, it is possible to enlarge this concept of liaison to curves having some common components, but I will not explain it here.

Now this relation is reflexive and symmetric, but not transitive, it generates an equivalence relation called *liaison* or *linkage*.

Definition Two curves C and C' are in the same liaison (resp. biliaison) class if and only if there exist an integer n and a sequence $C_0 = C, C_1, \ldots, C_n = C'$, such that C_i and C_{i+1} are linked by some complete intersection (resp. and n is even).

This concept has been introduced first by Apery ([A1], [A2]) and Gaeta ([G]), and developped by Peskine and Szpiro ([PS]). It turns out that it is a very useful tool in the classification of space curves. There are precise relations between the numerical invariants of the curves (degree, genus, dimension of the cohomology spaces) and we will see further that it is possible to characterize the biliaison classes. But, even if a curve is smooth, a linked curve can be non smooth. So, even if one is interested in smooth curves, it is necessary to study more general *locally Cohen-Macaulay* (l.C-M for short) curves. It is not necessary to give here a precise definition, but these curves have two important properties:

– i) the complete intersections are l.C-M,

– ii) if two curves C and C' are linked, C is l.C-M if and only if C' is l.C-M.

Therefore it is the natural context for liaison problems.

3. Functions on smooth space curves. Hartshorne-Rao module

Let f be a polynomial in 3 variables x_1, x_2, x_3. It defines in a natural way a function from the open subset of C where $X_0 \neq 0$ to $\mathbb{C}$. Morover, if f is one of the f_i, or more generally, if f belongs to the ideal $(f_1, \ldots, f_r)$ generated by the f_i, the corresponding function is the zero function. So we obtain a map, which is a ring homomorphism, from the quotient ring $A = \mathbb{C}[x_1, x_2, x_3]/(f_1, \ldots, f_r)$ to the ring of functions on C. One can prove that, if C is smooth, this map is injective. So A can be identified with a subring of the ring of functions on C. Elements of A are called *rational algebraic functions on* C, defined where $X_0 \neq 0$.

Suppose for simplicity that the plane "at infinity" $X_0 = 0$ intersects transversally the curves in d distinct points $P_1, \ldots, P_d$. If f is a non-zero element of A, we want to describe the corresponding function in a neighbourood of the P_i. For that purpose, we have the notion of "order" which is defined in the following way :

Let $f(\frac{X_1}{X_0}, \frac{X_2}{X_0}, \frac{X_3}{X_0}) = F(X_0, X_1, X_2, X_3)/X_0^n$, where F is an homogenous polynomial of degree n. It is the equation of a surface S, and we can define the "multiplicity of intersection" $m(C, S, P_i)$ of C and S at P_i (it is rather complicate, but it is 0 if S doesn't go through P_i, 1 if C and S intersect transversally at P_i, and ≥ 2 if S is tangent to C at P_i). Then the order of f at P_i is the number $ord_{P_i} f = m(C, S, P_i) - n$. If this order is positive (resp. negative), it means that f can be defined (resp. cannot be defined, we say that P_i is a pole of f) at P_i. With this definition, we see that $ord_{P_i} f \geq -n$.

Example : let C be the curve defined by the polynomials $(X_1^2 - X_2^2 - X_1X_0, X_3)$. It has 2 points at infinity, $P_1 = (0, 1, 1)$, $P_2 = (0, 1, -1)$. Let f be the function defined by the polynomial $x_1 - x_2 + \lambda$. Then one has $F = (X_1 - X_2 + \lambda X_0)/X_0$ and $ord_{P_i} f = m(C, S, P_i) - 1$. Since S doesn't go through P_2, we have $ord_{P_2} f = -1$. If $\lambda \neq -1/2$ (resp. $\lambda = -1/2$), S is transversal (resp. tangent) to C at P_1, so we have $ord_{P_1} f = 0$ (resp. $ord_{P_1} f = +1$). In any case, P_2 is a pole of f, but f is defined at P_1.

Linear systems on smooth space curves.

Thanks to this notion of order, we can introduce a filtration on the ring A by setting, for $n \in \mathbb{N}$,

$$A_n = \{f \in A \mid f = 0 \text{ or } ord_{P_i} f \geq -n \ \forall i = 1, \ldots d\}.$$

This is a finite dimensional vector space over $\mathbf{C}$, which is called by the algebraic geometers, for reasons that I can't develop here, the space of sections of the line bundle $\mathcal{O}_C(n)$. It is possible to define it in an intrinsic way.

For example, for $n = 0$, we obtain the set of *global functions* on the curve (functions defined everywhere).

For every positive n, recall that S_n is the set of homogenous polynomials of degree n; there exists a natural map of vector spaces $\phi_n : S_n \to A_n$ defined in the following way: if $F \in S_n$ is non-zero, $\phi_n(F)$ is the image of $f = F(1, x_1, x_2, x_3)$ in the quotient ring A (we have already seen that if $F \in S_n$, the order of f at P_i is $\geq -n$).

Moreover, we can put in a natural way a structure of graded S-algebra on the direct sum $A_C = \oplus_{n\in\mathbb{N}} A_n$, so that the direct sum of all the ϕ_n gives an homomorphism ϕ_C of graded S-algebra from S to A_C. Of course this map is not injective (every F_i defining the curve goes to 0), but we will see now that it can also be not surjective. However,

– if n is $>> 0$, or if n is $<< 0$, then ϕ_n is surjective,

– if C is a complete intersection (defined by two polynomials F and G), then ϕ_C is surjective (and $A_C = S/(F,G)$).

Definition The **Hartshorne-Rao module**, or Rao-module, M_C of a curve C is the cokernel of the map ϕ_C. It is a graded S-module of finite length (because it has only a finite number of non-zero homogenous components).

This module, which was introduced by Hartshorne and studied by Rao has very nice properties, and in some sense, it reflects algebraic properties of the curve. Its non-zero elements correspond to functions on the curve, which not come from functions defined on the projective space.

Example : let C be the curve defined by the polynomials $(Y_0X_2, Y_0X_3, X_1X_2, X_1X_3)$, where Y_0 is a linear form, independant of X_1, X_2, X_3, and not a multiple of X_0. It is the union of the two lines, L (defined by (Y_0, X_1)) and L' (defined by (X_2, X_3)) which don't intersect. There are two points at infinity, P on L and P' on L'.

Since the two lines are disjoint, one proves easily that the ring of rational algebraic functions on C (defined where $X_0 \neq 0$) is the product of the two corresponding rings of functions on L and L'. Therefore, A_C is in a natural way, the product of $A_L = S/(Y_0, X_1)$ and $A_{L'} = S/(X_2, X_3)$. The map $\phi_C : S \to S/(Y_0, X_1) \times S/(X_2, X_3)$ is the product of the two natural projections. Hence the Rao module M_C is $S/((Y_0, X_1, X_2, X_3)$. It has only one non-zero component, of dimension 1 and in degree zero., which correspond to the function taking the value 1 on one of the lines and 0 on the other one (there are two such functions, but their sum can be lifted on the projective space).

More generally, if C is the disjoint union of two c.i. (F_1, F_2) and (F_3, F_4), one proves that $M_C = S/(F_1, F_2, F_3, F_4)$.

4. Liaison and Rao module

There are nice connexions between these two notions, and they are summarized in the following results :

1. Two curves C and C' are in the same biliaison class if and only if there exists $h \in \mathbb{Z}$ such that $M_C \simeq M_{C'}(h)$ ([R]) (if $M = \oplus M_n$ is a graded S-module, $M(h)$ is the graded S-module defined by $M(h)_n = M_{n+h}$, we say that the degrees are shifted by h to the left).

   Example : the curves C with $M_C = 0$ are in the same biliaison class (which contains all the c.i., but not only c.i., for example the twisted cubics).

2. Let M be a graded S-module of finite length. There exist a smooth curve C and an integer $h \in \mathbb{Z}$ such that $M \simeq M_C(h)$ ([R]).

   Hence the Rao-modules (up to a shift) characterize the biliaison classes.

3. It is easy to prove, by an elementary geometric construction, that, starting from an existing Rao module, every right shift can be obtained. But there exists a minimal left shift ([Mi]). A corresponding curve is called a minimal curve in the biliaison class.

4. One knows how to construct explicitely the minimal curves from the module associated with the biliaison class ([MD-P2]).

5. One knows how to obtain the curves in the biliaison class from a minimal one ([BBM],[MD-P2]).

So the classification of these modules, which is an algebraic problem, will help us for the classification of space curves.

References

[A1] R. Apéry , Sur certains caractères numfiques d'un idéal sans composant impropre, C.R. Acad. Sci. Paris, t. 220, p. 234–236, 1945.

[A2] R. Apéry , Sur les courbes de première espèce de l'espace à trois dimensions, C.R. Acad. Sci. Paris, t. 220, p. 271–272, 1945.

[B] G. Bolondi, Irreducible families of curves with fixed cohomology, Arch. der Math., 53, 1989, 300–305.

[BB] E. Ballico et G. Bolondi, The variety of module structures, Arch. der Math. 54, 1990, 397–408.

[BBM] E. Ballico, G. Bolondi et J. Migliore, The Lazarsfeld-Rao problem for liaison classes of two-codimensional subschemes of $\mathbb{P}^n$, à paraître in Amer. J. of Maths.

[BM1] G. Bolondi et J. Migliore, Classification of maximal rank curves in the liaison class L_n, Math. Ann., 277, 1987, 585–603.

[BM2] G. Bolondi et J. Migliore, Buchsbaum Liaison classes, J. of Algebra, 123, 1989, 2, 426–456.

[Ein] L. Ein, Hilbert scheme of smooth space curves, Ann. Scient. E.N.S., 4^e série, 4, 1986, 469–478.

[E] G. Ellingsrud, Sur le schéma de Hilbert des variétés de codimension 2 dans $\mathbb{P}^e$ à cône de Cohen-Macaulay, Ann. Scient. E.N.S., 4^e série, 8, 1975, 423–432.

[G] F. Gaeta, Quelques progrès récents dans la classification des variétés algébriques d'un espace projectif, Liège, C.B.R.M. 145–181, 1952.

[GP1] L. Gruson et C. Peskine, Genre des courbes de l'espace projectif, Lecture Notes 687, Springer Verlag, 1977, 31–59.

[GP2] L. Gruson et C. Peskine, Genre des courbes de l'espace projectif (II), Ann. Scient. E.N.S., $4'^e$ série, 15, 1982, 401–418.

[H] G. Halphen, Mémoire sur la classification des courbes gauches algébriques, Oeuvres complètes, t.3, 261–455.

[jH] J. Harris, Curves in projective space, Presses de l'Université de Montréal, 1982.

[rH1] R. Hartshorne, On the classification of algebraic space curves, in Vector bundles and differential equations, Proceedings, Nice, France, 1979, Progress in Math.7, Birkhäuser.

[rH2] R. Hartshorne, Algebraic Geometry, 1977, Springer Verlag.

[LR] R. Lazarsfeld et A. P. Rao, Linkage of general curves of large degree, Lecture Notes 997, Springer Verlag, 1983, 267–289.

[MD-P1] M. Martin-Deschamps et D. Perrin, Sur la classification des courbes gauches I, C.R. Acad. Sci. Paris, t. 309, Série I, p. 345–350, 1989.

[MD-P2] M. Martin-Deschamps et D. Perrin, Sur la classification des courbes gauches Astérisque 184-185, 1990.

[Mi] J. Migliore, Geometric Invariants of Liaison, J. of Algebra, 99, (1986), 548–572.

[PS] C. Peskine et L. Szpiro, Liaison des variétés algébriques, Invent. Math., 26, 1974, 271–302.

[R] A. P. Rao, Liaison among curves in $\mathbb{P}^3$, Invent. Math., 50, 1979, 205–217.

Proc. of the 7^{th} EWM meeting, Madrid, 1995

On classification of projective varieties of small codimension

Emilia Mezzetti

Università di Trieste, Italy
mezzette@univ.trieste.it

1. Introduction

The subject of my paper is the "classification of embedded varieties", especially in low codimension. To clarify the meaning to be given to this term, let me start with the following theorem, which is classical:

Theorem 1 *Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Then X can be isomorphically projected in $\mathbb{P}^{2n+1}$.*

Idea of proof: the projection centered at a point P is an isomorphism from X to its image if and only if P does not belong to any secant or tangent line to X. The secant variety of X, $SecX$, which is the union of all secant and tangent lines to X, has dimension at most $2n+1$.

So if X is a curve, "natural setting" is $\mathbb{P}^3$, for surfaces $\mathbb{P}^5$, and so on. In other words, in $\mathbb{P}^3$ we find *all* curves, up to isomorphism. So to study the isomorphism classes of smooth curves (the *moduli space*) it is enough to study those of curves lying in $\mathbb{P}^3$. This was the subject of the paper by Mireille Martin-Descamps.

If X may be embedded in a projective space of dimension $< 2n+1$, then X is *in some sense* special. For example: plane curves, surfaces in $\mathbb{P}^3$ and $\mathbb{P}^4, \ldots$. We will try to understand in what sense X is special. Moreover there is the natural problem of the classification of such special varieties.

Recently there has been new interest in this subject, due particularly to a conjecture ("Hartshorne's conjecture") and a theorem (by Ellingsrud – Peskine). Moreover new techniques (as for example adjunction theorems and computational methods) have led to considerable progress.

2. Preliminary facts.

The first important observation is that the varieties of dimension n lying in $\mathbb{P}^{n+1}$ (i.e. of codimension 1) are precisely the *hypersurfaces*: they are sets of zeroes of a *unique* homogeneous polynomial in $n+2$ variables. Moreover, for any degree d, a general polynomial of degree d defines a smooth hypersurface.

We may say that, in codimension one, "special" means: only one equation is needed.

When we take the intersection of two hypersurfaces, then we get a variety which is *not necessarily irreducible.*

A simple example in $\mathbb{P}^3$: take the irreducible quadrics of equations

$$x_0x_2 - x_1^2 = 0, \quad x_0x_3 - x_1x_2 = 0.$$

The intersection contains the line $x_0 = x_1 = 0$; the other component is the skew cubic X, given parametrically by:

$$x_0 = s^3, \quad x_1 = s^2t, \quad x_2 = st^2, \quad x_3 = t^3.$$

So the two components are curves intersecting at the point $(0:0:0:1)$. If we intersect further with the quadric

$$x_1x_3 - x_2^2 = 0,$$

then the dimension does not decrease: we find precisely X.

In general, we have the following:

Theorem 2 *Let $X, Y \subset \mathbb{P}^m$ have dimension r, s respectively. Let Z be an irreducible component of $X \cap Y$. Then $dim\ Z \geq r + s - m$.*

In particular, if $X_1, X_2, ..., X_r$ are hypersurfaces in $\mathbb{P}^m$ and Z is an irreducible component of their intersection, then $dim\ Z \geq m - r$. Note that in the previous example the inequality was strict.

Now we can define the varieties complete intersection: X of dimension n in $\mathbb{P}^m$ is a *complete intersection* (c.i. for short) if X is the *transversal* intersection of $m-n$ hypersurfaces, or, equivalently, the homogeneous ideal of X, $I(X)$, is generated by $m - n$ homogeneous polynomials.

It is possible to prove that a general c.i. variety is smooth. The canonical sheaf of a c.i. X of type $r_1, ..., r_s$ in $\mathbb{P}^m$ is $\mathcal{O}_X(r_1 + ... + r_s - m - 1)$; this implies that, except for a few particular cases, the c.i. are varieties of general type.

By Bezout's theorem, if X is a c.i. such that

$$I(X) = < F_1, ..., F_r >, \quad r = m - n,$$

then the degree of X satisfies

$$degX = (degF_1)(degF_2)...(degF_r)$$

.

3. Hartshorne's conjecture (1974).

It says the following [H]:

Let $X \subset \mathbb{P}^m$ be a smooth variety of dimension n. If $n > \frac{2m}{3}$, then X is a complete intersection.

The assumption is that the codimension of X is "small enough" with respect to the dimension, i.e. *codim* $X := m - n < \frac{n}{2}$. So, in this range, to be "special" should mean to be a complete intersection.

The first significant case is in $\mathbb{P}^7$ with codimension 2. It is easy to construct examples of non c.i. smooth varieties of small dimension. For example: any skew cubic curve in $\mathbb{P}^3$ is not c.i. by Bezout, because its degree is prime; any curve in $\mathbb{P}^3$ with Hartshorne – Rao module not 0 is not c.i.. Moreover, cones over non c.i. curves provide examples of *singular* non c.i. varieties in the range of the conjecture.

Support to the Hartshorne's conjecture:

– lack of examples;

– a theorem of Barth ([B] 1970), which shows that, from a topological point of view, smooth varieties of small codimension are similar to complete intersections.

Precisely, by the theorem of Lefschetz, if X is c.i. in $\mathbb{P}^m$, then the restriction maps

$$H^i(\mathbb{P}^m, \mathbb{C}) \to H^i(X, \mathbb{C})$$

are isomorphisms for $i \leq n-1$. The same over $\mathbb{Z}$.

Barth's theorem: if X is smooth, then the analogous restriction maps are isomorphisms for $i \leq 2n - m = n - (m-n)$. (The same over $\mathbb{Z}$ was successively proven by Larsen [L]).

Hence the more the codimension is small, the more the cohomology is similar to that of complete intersections.

Two classical examples show that the bound in the conjecture is sharp. We have the following non c.i. varieties with $3n = 2m$:

(a) $\mathbb{G}(1,4) \subset \mathbb{P}^9$, of dimension 6, degree 5, non-degenerate (Bezout!);

(b) the spinor variety $S \subset \mathbb{P}^{15}$, of dimension 10, parametrizing the $4-$planes of a family in a $8-$dimensional quadric.

No examples in $\mathbb{P}^6, \mathbb{P}^{12}$, etc. (Zak [Z]: these are the unique manifolds with $3n = 2m$, non c.i. and corresponding to orbits of linear algebraic groups).

For codimension 2 varieties, the range of the Hartshorne's conjecture is from $\mathbb{P}^7$ on. But there are no examples already in $\mathbb{P}^6$.

Another interpretation of the conjecture is as follows: From Barth–Larsen, it follows that the smooth varieties X of codimension 2 in $\mathbb{P}^n$ are subcanonical for $n \geq 6$, i.e. the canonical bundle is $\omega_X = \mathcal{O}_X(k)$, for some integer k. By the Serre correspondence ([OSS]), each subcanonical X is the zero locus of a section of an algebraic vector bundle E of rank 2. Such a X is c.i. if and only if E is decomposable: $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$. So the Hartshorne's conjecture for codimension 2 varieties becomes: "if $n \geq 7$, then each algebraic vector bundle of rank 2 is of the form $\mathcal{O}(a) \oplus \mathcal{O}(b)$". In fact, there are no examples of indecomposable rank two bundles already for $n \geq 5$.

Some partial results about Hartshorne's conjecture:

- for small degree: the non-necessarily smooth varieties of small degree are all classified (Weil [X] for $d = 3$, Swinnerton-Dyer [SD] for $d = 4$, Ionescu [Io1],[Io2]) and many others for successive degrees);

- concerning $k-$normality: by definition X is $k-$normal if the hypersurfaces of degree k cut on X a complete linear series; c.i. are $k-$normal for any k;

- there exists a function $N(d)$ of the degree, such that each non c.i. smooth X of degree d is contained in $\mathbb{P}^{N(d)}$ ([H]). The best known estimates are:

- $N(d) \sim \frac{5}{2}d^2$ for any codimension (Barth-Van de Ven [BV]);
- $N(d) \sim \sqrt{d}$ for codimension 2 (Holme-Schneider [HS]);
- $N(d) \leq 2rd - d$, for codimension r (Bertram-Ein-Lazarsfeld [BEL]).

4. Varieties of codimension 2 in $\mathbb{P}^4$ and $\mathbb{P}^5$.

We consider now varieties of small codimension but out of the range of Hartshorne's conjecture.

Theorem 3 (Ellingsrud-Peskine [EP], 1989) *There is a finite number of families of smooth surfaces of $\mathbb{P}^4$ not of general type.*

Here "family" means "irreducible component of the Hilbert scheme of surfaces of $\mathbb{P}^4$".

Some word about the Hilbert scheme: if X is a projective variety in $\mathbb{P}^m$ and $I(X)$ is its homogeneous ideal, then the quotient ring

$$S(X) := \frac{\mathbb{C}[x_0, ..., x_m]}{I(X)}$$

is naturally graded and its homogeneous elements of degree d can be interpreted as hypersurfaces of degree d not containing X. The Hilbert function of X is defined by: $h_X(t) = dim S(X)_d$ as a $\mathbb{C}$-vector space. There is a unique numerical polynomial $P_X(t)$, the Hilbert polynomial of X, such that $P_X(t) = h_X(t)$ for $t >> 0$, t integer. The coefficients of P_X have important geometrical meaning: they give the dimension of X, the degree, the Euler-Poincaré characteristic χ, the sectional genus of X i.e. the genus of a curve intersection of X with a general linear space of the right dimension, etc.

An important theorem of Grothendieck ([G]) says that: fixed any numerical polynomial P, the set of subschemes of $\mathbb{P}^m$ having P as Hilbert polynomial has a natural structure of projective scheme, enjoying a nice universal property.

For example, for curves, fixing P means fixing degree and genus; for surfaces degree, χ and sectional genus, etc.

So in particular the theorem says that there is an upper bound d_0 on the degree of non-general type surfaces lying in $\mathbb{P}^4$.

Braun-Fløystad [BF](1993): $d_0 \leq 105$. Then M. Cook ([C]) gave a better bound. But all known examples have $d \leq 15$.

Analogous result for $3-$dimensional manifolds in $\mathbb{P}^5$ was proven by Braun-Ottaviani-Schneider-Schreyer ([BOSS] 1993). In this case there are examples up to degree 18.

After the theorem of Ellingsrud - Peskine, the problem arises of giving a list of *all* families of surfaces of $\mathbb{P}^4$ not of general type; in particular rational and (birationally) ruled surfaces. Similar problem for $3-$folds in $\mathbb{P}^5$. This has been made for small degrees (Okonek [O1], [O2],[O3], Ranestad [R], Aure-Ranestad [AR], Popescu [P] for surfaces, Beltrametti-Schneider-Sommese [BSS] for threefolds). I would like to mention two important methods:

a) **by the adjunction map**. This is a classical method for studying algebraic surfaces, which goes back to Castelnuovo and Enriques. Given X, smooth connected variety of dimension n, and a very ample divisor H on X corresponding to a line bundle L, one considers *the adjoint linear system*, i.e. $\mid K_X + (n-1)H \mid$, where K_X is the canonical divisor.

Sommese-Van de Ven ([S] and [V],1979) have characterized the varieties X such that the adjoint linear system is not base-point-free, or, equivalently, the line bundle $\omega_X \otimes L^{\otimes(n-1)}$ is not generated by global sections.

They are: $\mathbb{P}^n$ with $L = \mathcal{O}(1)$, $\mathbb{P}^2$ with $L = \mathcal{O}(2)$ (i.e. the Veronese surface), smooth quadrics, scrolls over a smooth curve i.e. varieties X which are ruled by linear spaces of dimension $n-1$ over a smooth curve. For any other X, there exists the (regular) adjunction mapping $\phi := \phi_{|K_X+(n-1)H|}$. The next step is the classification of the varieties X such that $\phi(X)$ has dimension $< n$ (Fano varieties, quadric bundles, scrolls over a surface) or is not birational (there are four examples). For the remaining varieties, ϕ contracts exceptional divisors contained in X.

For surfaces: by iteration of this procedure, one gets either a minimal model or a surface for which the adjunction map is not defined or not birational. For example, for rational surfaces of low degree in $\mathbb{P}^4$, one can explicitly find a linear system of plane curves defining it. For threefolds, it is necessary to study also the linear system $\mid K_X + H \mid$ and possibly its multiples.

b) **the computational method** by Decker-Ein-Schreyer ([DES]). The ideal sheaf of a codimension 2 variety X in $\mathbb{P}^m$ has a locally free resolution of the form:

$$O \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{I}_X(k) \longrightarrow 0$$

for some integer k and locally free sheaves $\mathcal{F}, \mathcal{G}$ on $\mathbb{P}^m$ of ranks $f, f+1$. The idea is: try to construct X starting from $\mathcal{F}$ and $\mathcal{G}$. Taking the cohomology table of $\mathcal{I}_X$ (assuming that X exists), one looks for sheaves $\mathcal{F}$ and $\mathcal{G}$ with the "right cohomology". Then one takes a map $\phi : \mathcal{F} \to \mathcal{G}$ and checks (with a computer) if the degeneracy locus is smooth. By this method, it has been possible to refind all known examples of surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$.

5. Related problems.

Study the geometry of varieties which are not general. For example, the Veronese surface in $\mathbb{P}^4$. It is a general projection of the Veronese surface V of $\mathbb{P}^5$, i.e. the plane embedded via the complete linear system of the conics. This projection is isomorphic because $Sec\ V$ is a hypersurface.

(i) The theorem of Severi ([fS] 1901) says that all smooth surfaces of $\mathbb{P}^4$, except the Veronese surface, are linearly normal, i.e. they are not isomorphic projections of surfaces contained in spaces of higher dimension. In fact, if $X \subset \mathbb{P}^5$, X not the Veronese surface, then $SecX = \mathbb{P}^5$. The smooth 3−folds of $\mathbb{P}^5$ are all linearly normal (by a theorem of Zak [Z]). There is a unique example of a 3−fold X which is not 2−normal: this means that the quadrics of $\mathbb{P}^5$ don't cut a *complete* linear series on X. This example is called the Palatini scroll; it is ruled by lines over a cubic surface of $\mathbb{P}^3$. Its ideal has a resolution as follows:

$$0 \to \mathcal{O}^4_{\mathbb{P}^5} \to \Omega^1_{\mathbb{P}^5}(2) \to \mathcal{I}_X(4) \to 0 \qquad (*)$$

so $h^1(\mathcal{I}_X(2)) = 1$, $h^1(\mathcal{I}_X(k)) = 0$ for $k \neq 1$.

There is a strict analogy of X with the Veronese surface of $\mathbb{P}^4$ S because $\mathcal{I}_S$ has a very similar resolution:

$$0 \to \mathcal{O}^3_{\mathbb{P}^4} \to \Omega^1_{\mathbb{P}^4}(2) \to \mathcal{I}_S(3) \to 0$$

so $h^1(\mathcal{I}_S(1)) = 1$.

Conjecture (Peskine, Van de Ven): X is the unique non 2−normal 3−fold of $\mathbb{P}^5$.

(ii) A classical theorem of C. Segre ([cS] 1921) states that, if a smooth surface S of $\mathbb{P}^4$ contains a 2−dimensional family of plane irreducible curves, then these curves are conics and S is a projection of the Veronese surface of $\mathbb{P}^5$, hence a Veronese surface of $\mathbb{P}^4$ or a cubic scroll. It is possible to prove ([M]) that if a smooth 3−fold X of $\mathbb{P}^5$ contains a family of dimension 3 of plane curves, then either X is contained in a quadric or the degree of these curves is at most 3. Let us assume that X is not contained in a quadric: then ([MP] there are only two 3−folds satisfying this assumption: the Palatini scroll again and the Bordiga scroll: this last one may be obtained by an exact sequence like (*), by a particular bundle map.

(iii) A classical way for studying projective manifolds of dimension n is to study their projections into $\mathbb{P}^{n+1}$. A theorem of Franchetta ([F] 1941) says that, if $S \subset \mathbb{P}^4$ is not the Veronese surface, then the double locus of its general projection in $\mathbb{P}^3$ is an irreducible curve. For 3−folds in $\mathbb{P}^5$, the general projection in $\mathbb{P}^4$ has always an irreducible surface as double locus. The triple locus is a curve and, if X is a Palatini scroll, then this curve is reducible: this is the unique known example.

Work in progress of Mezzetti and Portelli is related to the points **(i)**, **(ii)**, **(iii)**.

References

Basic books (here one can find the classical theorems quoted in this paper).

P. Griffiths - J. Harris, "Principles of Algebraic Geometry", J. Wiley & Sons, 1978;

R. Hartshorne, "Algebraic Geometry", Springer, 1977;

I. Shafarevich, "Basic Algebraic Geometry", Springer, 1974;

Articles.

[AR] A.B. Aure - K. Ranestad, *The smooth surfaces of degree* 9 *in* $\mathbb{P}^4$, London Math. Soc. LNS, 179 (1992), 32-46;

[B] W. Barth, *Transplanting cohomology classes in complex - projective space*, Amer. J. Mat. **92** (1970), 951-967;

[BV] W. Barth - A. Van de Ven, *A decomposability criterion for algebraic* 2−*bun- dles*, Invent. Math. **25** (1974), 91-106;

[BSS] M. Beltrametti - M. Schneider - A. Sommese, *Threefolds of degree* 9 *and* 10 *in* $\mathbb{P}^5$, Math. Ann. **288** (1990), 413-444

[BEL] A. Bertram - L. Ein - R. Lazarsfeld, *Vanishing theorems, a theorem of Severi, and the equations defining projective varieties*, Journal Amer. Math. Soc. **4** (1991), 587-602

[BF] R. Braun - G. Floystad, *A bound for the degree of smooth surfaces in* $\mathbb{P}^4$ *not of general type*, Compositio Math. **93** (1994), 211-229

[BOSS] R. Braun - G. Ottaviani - M. Schneider - F. Schreyer, *Boundedness for non general type* 3−*folds in* $\mathbb{P}^5$, to appear

[C] M. Cook, *An improved bound for the degree of smooth surfaces in* $\mathbb{P}^4$ *not of general type*, preprint (1995)

[DES] W. Decker - L. Ein - F. Schreyer, *Construction of surfaces in* $\mathbb{P}^4$, J. Alg. Geometry **2** (1993), 185-237

[EP] G. Ellingsrud - Ch. Peskine, *Sur les surfaces lisses de* $\mathbb{P}_4$, Invent. Math. **95** (1989), 1-12;

[F2] A. Franchetta, *Sulla curva doppia della proiezione della superficie generale dell'*S_4*, da un punto generico su un* S_3, Lincei - Rend. Sc. fis. mat. e nat., Vol. II (1947), 276-279;

[G] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géo- métrie algébrique, IV: Les schémas de Hilbert*, Séminaire Bourbaki, exp. 221 (1960-61);

[H] R. Hartshorne, *Varieties of small codimension in projective space*, Bull.Amer. Math. Soc. **80** (1974), 1017-1032;

[HS] A. Holme - M. Schneider, *A computer aided approach to codimension two subvarieties of* $\mathbb{P}^n$*,* $n \geq 6$, Crelle J. **357** (1985), 205-220;

[Io1] P. Ionescu, *Embedded projective varieties of small invariants*, Lect. Notes in Math. 1056 (1984), 142-186;

[Io2] P. Ionescu, *Embedded projective varieties of small invariants II*, Rev. Roum. Math. Pures Appl. **31** (1986), 539-544;

[L] M. E. Larsen, *On the topology of complex projective manifolds*, Invent. Math. **19** (1973), 251-260;

[M] E. Mezzetti, *Projective varieties with many degenerate subvarieties*, BUMI (7) 8-B (1994), 807-832;

[MP] E. Mezzetti - D. Portelli, *Threefolds in* $\mathbb{P}^5$ *with a* $3-$*dimensional family of plane curves*, preprint (1995);

[O1] C. Okonek, *Moduli reflexiver Garben und Flächen vom kleinem Grad in* $\mathbb{P}^4$, Math. Z. **184** (1983), 549-572;

[O2] C. Okonek, *Über* $2-$*codimensionale Untermannigfaltigkeiten vom Grad* 7 *in* $\mathbb{P}^4$ *und* $\mathbb{P}^5$, Math. Z. **187** (1984), 209-219;

[O3] C. Okonek, *Flächen vom Grad* 8 *in* $\mathbb{P}^4$, Math. Z. **191** (1986), 207-223;

[OSS] C. Okonek - M. Schneider - H. Spindler, "Vector Bundles on Complex Projective Spaces", Birkhauser (1980)

[P] S. Popescu, *On smooth surfaces of degree* ≥ 11 *in* $\mathbb{P}^4$, Dissertation, Univ. Saarbrucken (1993);

[R] K. Ranestad, *On smooth surfaces of degree ten in the projective four space*, Thesis, Univ. Oslo (1988);

[cS] C. Segre, *Le superficie degli iperspazi con una doppia infinità di curve piane o spaziali*, Atti R. Acc. Scienze di Torino, Vol. LVI (1921), 75-89; also in *Opere*, Vol. II, XXXIII, 163-175;

[fS] F. Severi, *Intorno ai punti doppi improprii di una superficie generale dello spazio a quattro dimensioni, e ai suoi punti tripli apparenti*, Rend. Circolo Matematico di Palermo, **15** (1901), 33-51; also in *Opere Matematiche*, Vol. I, VI, 14-30;

[aS] A. Sommese, *Hyperplane sections of projective surfaces I. The adjunction mapping*, Duke Math. J. **46** (1979), 377-401;

[SD] H. P. F. Swinnerton-Dyer, *An enumeration of all varieties of degree* 4, Amer. J. Math. **95** (1973), 403-418;

[V] A. Van de Ven, *On the* $2-$*connectedness of very ample divisors on a surface* Duke Math. J. **46** (1979), 403-407;

[Z] F. L. Zak, *Tangents and Secants of Algebraic varieties*, Translations of Mathematical Monographs 127, AMS (1993)

[X] XXX, *Correspondence*, Amer. J. Math. **79** (1957) 951-952

Mathematical Physics

Session organised by Sylvie Paycha

The aim of the session organised on **mathematical techniques in statistical physics and quantum field theory** was to give an idea of some of the mathematical problems that have recently arisen in relation to both these fields. Because of the wide spectrum of the topics presented within this session, the speakers were asked not so much to go into technical details but rather to give a brief overview of the mathematical techniques involved in their work and to try to show the motivations behind them arising from statistical physics of quantum field theory.

The topics of the talks were chosen in such a way that the spectrum of mathematical techniques involved would be as broad as possible; probability theory (Flora Koukiou), algebraic techniques (Marjorie Batchelor), analysis on super manifolds (Alice Rogers) and algebraic geometry (Claire Voisin). In this sense, the emphasis during this session was not put on a systematic presentation of the mathematical tools, but rather on how they can be used to understand problems of physical origin such as phase transition (Flora Koukiou), the structure of spaces of maps involved in the path space description of certain quantum field theories (Marjorie Batchelor), quantisation of supersymmetric theories (Alice Rogers) and some aspects of super conformal theories (Claire Voisin).

We hope these talks will give the reader some insight in these fields of mathematical physics and convince her/him that theoretical physicists and mathematicians still have a lot to learn from each other!

Sylvie Paycha

Proc. of the 7^{th} EWM meeting, Madrid, 1995

Mirror Symmetry

Claire Voisin (written by Sylvie Paycha and Alice Rogers)

Université de Paris XI, France
voisin@diam1.ens-cachan.fr

Warning : *This is a summary of two introductory talks given by Claire Voisin on mirror symmetry. These notes (revised by C. Voisin) were written by S. Paycha and A. Rogers following the lectures given by Claire Voisin. Because of the length of the talks, we could only offer here a summary of these lectures. However, we have tried to keep the spirit of Claire Voisin's talk in which her main concern was not so much to go into technical details as to give the audience an idea of the main topics of research related to mirror symmetry, a field which is evolving rapidly. We have also tried to make these notes accessible to non specialists, at the cost of remaining vague. The reader interested in further details is referred to Claire Voisin's book on Mirror Symmetry and references therein.*

Abstract: Calabi-Yau manifolds are defined and the notion of a mirror pair is introduced. The physical origins of mirror symmetry are described, together with steps towards a more mathematical understanding.

1. Introduction

This paper is devoted to the description of the mirror symmetry phenomena; first discovered by physicists, it should associate to a Calabi-Yau variety X a mirror X' satisfying $H^{p,q}(X) = H^{n-p,q}(X')$, $n = dim_{\mathbb{C}}X = dim_{\mathbb{C}}X'$. Furthermore, one should have an identification for the moduli space $\mathcal{M}_X$ parametrising marked complex structures plus complexified Kähler parameters on X with the corresponding moduli space $\mathcal{M}_{X'}$ on X', in such a way that the two factors are interchanged.

Physicists discovered mirror symmetry via the study of superconformal field theories derived from supersymmetric σ models on Calabi-Yau manifolds. Further information, with important mathematical applications, is obtained from the Yukawa couplings of these theories.

2. Yau's Theorem

In this paragraph, we recall some basic notions in complex analysis and algebraic geometry which occur in the description of mirror symmetry that follows (see e.g [GH]).

2.1. Kählerian manifolds

Let X be a differentiable manifold of dimension N and $\Omega^k(X, \mathbb{R})$ denote the space of real differential forms of degree k on X, the elements of which can locally be seen as linear combinations of expressions of the type $dx_{i_1}\wedge\cdots\wedge dx_{i_k}$ with coefficients given by differentiable functions on X, where $\{i_1,\cdots,i_k\}\subset\{1,\cdots,N\}$ when using local coordinates $(x_1,\cdots,x_N)$.

Using the exterior differentiation $d:\Omega^k(X,\mathbb{R})\to\Omega^{k+1}(X,\mathbb{R})$, one can define De Rham cohomology spaces $H^k(X,\mathbb{R})$ (resp. $H^k(X,\mathbb{C})$) as quotients Ker d/Imd of the space of real (resp.complex) closed k-forms (recall that α is closed whenever $d\alpha=0$) on X by the space of real (resp.complex) exact k-forms (recall that α is an exact form whenever there is a form β such that $\alpha=d\beta$).

Let now X be a complex manifold of dimension n and let $\Omega^{p,q}(X)$ denote the space of complex forms of type (p,q) on X, the elements of which can be seen as linear combinations of expressions of the type $dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\cdots\wedge d\bar z_{j_q}$ with coefficients given by differentiable functions on X (where $\{i_1,\cdots,i_p\},\{j_1,\cdots,j_q\}$ are subsets of $\{1,\cdots,n\}$) when using local coordinates $(z_1,\cdots,z_n)$.

In particular a complex two-form on X is an element of $\Omega_X^{2,0}\oplus\Omega_X^{1,1}\oplus\Omega_X^{0,2}$. More generally, we have

$$\Omega_X^k\equiv\sum_{p+q=k}\Omega_X^{p,q}.$$

Using the operator $\bar\partial:\Omega_X^{p,q}\to\Omega_X^{p,q+1}$, (defined by $\bar\partial=\sum_{i=1}^n d\bar z^i\partial/\partial\bar z^i$), one can define in a similar way as above the Dolbeault cohomology spaces $H^{p,q}(X)$ as quotients $Ker\bar\partial/Im\bar\partial$. Since for fixed p, $(\Omega_X^{p,q},\bar\partial)$ is a resolution of the holomorphic sheaf $\Omega_X^{p,0}$, we have $H^{p,q}(X)=H^q(\Omega_X^{p,0})$.

A hermitian metric h on a complex manifold X is a hermitian structure h_x on each tangent space T_xX which varies in a differentiable way with x.

A Kähler manifold is a complex manifold X which can be equipped with a hermitian structure, the imaginary part of which is a closed form (called the Kähler form) of type $(1,1)$ that is J-invariant, in other words, such that $h=g-i\omega$, with $d\omega=0$, g (resp. ω) being a symmetric (resp. antisymmetric) real form. It can also be seen as a complex manifold X equipped with a hermitian metric h such that the Levi-Civita connexion ∇ for the metric $g=\Re(h)$ is compatible with the hermitian structure, i.e $\nabla=\nabla^{1,0}\oplus\nabla^{0,1}$ and $dh(u,v)=h(\nabla^{1,0}u,v)+h(u,\nabla^{0,1}v)$.

Let us now assume X is compact and Kählerian. There is a Hodge decomposition for the cohomology spaces $H^k(X,\mathbb{C})$ into a direct sum of Dolbeaut cohomology spaces. The cohomology spaces $H^k(X,\mathbb{C})$ and $H^{p,q}(X)$ are related by:

$$H^k(X,\mathbb{C})=\oplus_{p+q=k}H^{p,q}(X).$$

Since a Kähler form ω is a real closed form, one can consider its Kähler class $[\omega]$ in $H^2(X,\mathbb{R})$ and we have $[\omega]\in H^{1,1}(X)\cap H^2(X,\mathbb{R})$, because ω is everywhere of type $(1,1)$.

Lemma 1 *The set of Kähler classes in $H^2(X,\mathbb{R})$ is an open cone - the Kähler cone- in $H^{1,1}(X)\cap H^2(X,\mathbb{R})$.*

Remark 2 In the case when $H^{1,1}(X)=H^2(X,\mathbb{C})$, the Kähler cone is open in $H^2(X,\mathbb{R})$ and hence contains integer classes. X is then algebraic by Kodaira's theorem.

2.2. Calabi-Yau manifolds and Yau's theorem

Let X be a complex manifold and let T_X denote its tangent bundle (the fibre of which is the tangent space T_xX at point x), T_X^* the cotangent bundle (the fibre of which is the dual space T_x^*X to the tangent space at point x). Both these bundles split into a direct sum $T_X = T_X^{1,0} \oplus T_X^{0,1}$, $T_X^* = {T^*}_X^{1,0} \oplus {T^*}_X^{0,1}$, $T_X^{1,0}$ (resp.${T^*}_X^{1,0}$) corresponding to the holomorphic part of the tangent (resp. cotangent) bundle and $T_X^{0,1}$ (resp. ${T^*}_X^{0,1}$) to its antiholomorphic part.

The canonical line bundle K_X is the holomorphic vector bundle $\Lambda^n {T^*}_X^{1,0}$ of rank 1 on X, the fibre of which is generated by the n-form $dz_1 \wedge \cdots \wedge dz_n$ using holomorphic local coordinates $(z_1, \cdots, z_n)$. It is trivial whenever there is a nowhere vanishing holomorphic n-form on X.

A Calabi-Yau manifold is a compact Kähler manifold with trivial canonical line bundle. Elliptic curves, abelian surfaces are examples of Calabi-Yau manifolds. They are often obtained as hypersurfaces inside larger manifolfds, like quintics which are hypersurfaces of degree 5 (see [GH] for the notion of degree of a variety) in the projective space P^4.

Let $-K_X$ denote the vector bundle $\Lambda^n T^{1,0}X$ of rank 1 on X, the fibre of which is generated by $\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}$ using local coordinates $(z_1, \cdots, z_n)$. A hermitian metric h on X induces a hermitian structure on the tangent bundle T_X (which means a hermitian structure on each fibre T_xX, the hermitian structures depending in a differentiable way on the base point x) and hence a metric h_{-K} on $-K_X$. The Kähler metric h will be called Kähler-Einstein whenever the curvature

$$\omega_{-K} \equiv \frac{1}{2i\pi} \partial \bar{\partial} h_{-K}$$

of h_{-K} vanishes. This two form is related to the Ricci curvature of the Riemannian metric g underlying the hermitian structure h so these metrics are in fact Ricci flat.

Whereas given α in the Kähler cone of a given Kähler manifold, there are various hermitian metrics $h = g - i\omega$ such that $[\omega] = \alpha$, the following theorem by Yau shows that on a Calabi-Yau manifold, Ricci flat Kähler metrics are in one to one correspondance with cohomology classes of Kähler forms.

Theorem 3 (Yau) *On a Calabi-Yau manifold (X, h), for any α in the Kähler cone, there is a unique Ricci flat Kähler metric $h = g - i\omega$ on X, with Kähler form ω in the cohomology class $\alpha \in H^2(X, \mathbb{R})$ of α.*

3. Mirror symmetry

3.1. Deformations of complex structures

A complex structure on X is described in terms of a field of linear operators $J_x, x \in X$ each acting on the tangent space T_xX with the property that $J_x^2 = -I$ (this yields a pseudo-complex structure) and satisfying an integrability condition (which makes the pseudo-complex structure into a complex structure). The eigenspaces of J_x corresponding to the eigenvalues i and $-i$ yield a splitting $T_xX = T_x^{1,0}X \oplus T_x^{0,1}X$. A deformation of the complex structure is a deformation J_t of this field of operators J which yields another splitting

$$T_xX = T_{x,t}^{1,0}X \oplus T_{x,t}^{0,1}X.$$

Let us introduce the Dolbeault cohomology group defined as the quotient space

$$H^1(T_X) \equiv Ker(\bar{\partial} : T_X^{1,0} \otimes \Omega_X^{0,1} \to T_X^{1,0} \otimes \Omega_X^{0,2}) / Im(\bar{\partial} : T_X^{1,0} \to T_X^{1,0} \otimes \Omega_X^{0,1})$$

It is a general fact that $H^1(T_X)$ parametrizes first order deformations of the complex structures on X. An important fact is a result due to Bogomolov, Tian and Todorov and proved recently in a more algebraic way by Ran concerning deformations of complex structures on a Calabi-Yau manifold.

Theorem 4 *The small deformations of the complex structure J on a Calabi-Yau manifold form a smooth family -<u>the Kuranishi family</u>-whose tangent space at X is $H^1(T_X)$.*

Remark 5 Deformations of the complex structure do not modify the triviality of the canonical bundle K_X.

3.2. Moduli space

Although the cohomology group $H^{1,1}(X)$ in which lies the Kähler cone is naturally real, one is led to complexifying the Kähler cone, thus introducing a <u>complexified Kähler parameter</u> $\omega = \alpha + i\beta$, where α is in the Kähler cone and $\beta \in H^2(X, \mathbb{R})$ is defined modulo $2\pi H^2(X, \mathbb{Z})$.

For each complex structure J_t on X making X into a complex manifold X_t, we can define such a complexified Kähler parameter $\omega \in H^2(X_t, \mathbb{C})$ modulo $2i\pi H^2(X, \mathbb{Z})$. Assuming that the manifold X satisfies the condition $H^2(X, \mathbb{C}) = H^{1,1}(X)$, the Kähler cone is open in $H^2(X, \mathbb{R})$ and hence ω varies locally in an open set of $H^2(X_t, \mathbb{C})$. $H^2(X_t, \mathbb{C})$ being locally a constant vector space independent of t- this means that deformations of the complex structure $X \to X_t$ followed by deformations of the Kähler structure ω on X_t can be locally interpreted as products of complex deformations and Kähler deformations.

Let X be a Calabi-Yau manifold. The <u>moduli space</u> $\mathcal{M}_X \equiv \{(X_t, \omega)\}$ is the set of couples with first term given by an isomorphism class of complex structures X_t on X obtained by deformations J_t of the initial complex structure J on X and second term given by a complexified Kähler parameter ω on X_t. From the above discussion it follows that if X satisfies the condition $H^2(X, \mathbb{C}) = H^{1,1}(X)$, then $\mathcal{M}_X$ is locally a direct product. However, this product structure does not hold in general globally, in particular since the Kähler cone can depend on t.

This local product structure reads on the level of tangent spaces as:

$$T\mathcal{M}_X = H^1(T_X) \oplus H^1(\Omega_X)$$

since $H^1(T_X)$ describes infinitesimal deformations of complex structures and $H^1(\Omega_X)$ describes infinitesimal deformations of Kähler strutures.

3.3. Mirror symmetry

Mirror symmetry predicts the existence for a given Calabi-Yau manifold X, of a Calabi-Yau <u>mirror</u> manifold X' of same dimension as X and such that there is an isomorphism $\mathcal{M} : \mathcal{M}_X \cong \mathcal{M}_{X'}$ by which complex deformations and Kähler deformations are swapped or in other words with the property that the local product structure of moduli space is preserved but the factors exchanged.

This "conjecture" predicted by physicits cannot be completely true because there are rigid Calabi-Yau manifolds which have no complex structure moduli, whose mirror could thus not even be Kählerian. Although it has been confirmed in a wide range of cases it is not yet mathematically fully understood.

Remark 6 The manifold X and its mirror X' are in general topologically very different!

The conjecture mathematically translates as follows. Since the local structure of $\mathcal{M}_X$ arises from the splitting

$$T\mathcal{M}_X = H^1(T_X) \oplus H^1(\Omega_X),$$

one expects the tangent map $\mathcal{M}_*$ to $\mathcal{M}$ to induce isomorphisms:

$$H^1(T_X) \simeq H^1(\Omega_{X'})$$

and

$$H^1(\Omega_X) \simeq H^1(T_{X'}).$$

More generally, one expects a sequence of isomorphisms:

$$H^p(\Lambda^q T_X) \simeq H^p(\Lambda^q \Omega_{X'})$$

Since the canonical line bundle K_X is trivial, the holomorphic bundles $\Lambda^q T_X$ and $\Lambda^{n-q}\Omega_X$ are isomorphic, using a global section of K_X and this induces isomorphis ms:

$$H^p(\Lambda^q T_X) \simeq H^p(\Lambda^{n-q}\Omega_X)$$

where n is the dimension of X and hence the Betti numbers $h^{p,q}(X') \equiv dim H^{p,q}(X')$ of X' (see [GH]) should be equal to $h^{n-p,q}(X)$ of X. Hence, the topology of X' is determined by that of X.

4. Mirror symmetry and physics

4.1. The N=2 supersymmetric σ-model

Let (X, g) be a Riemannian manifold. Just as point-particles are classically described by paths $\phi : [0, T] \to \mathbb{R}^d$ evolving in space-time which minimise the energy $S(T) \equiv \int_0^T \|\frac{d\phi}{dt}\|^2$ given in terms of their length, we shall define a bosonic σ- model classically by "paths" $\phi : \Sigma \to X$ defined as maps on a Riemann surface Σ with metric γ with values in M which minimise the action (or energy)

$$S(\phi, \gamma) \equiv \int_\Sigma \|d\phi\|^2.$$

S is invariant under conformal transformations of Σ, i.e under diffeomorphisms of Σ which multiply the metric γ pointwise by a positive factor, namely diffeomorphisms f such that there is a function h on Σ with $(f_*\gamma)(x) = e^h(x)\gamma(x)$. When X is Kählerian, and α is the Kähler form on X, the action reads

$$S(\phi, \gamma) = \int_\Sigma \phi^*\alpha + \int_\Sigma 2\|\bar{\partial}\phi\|^2$$

where $\phi^*\alpha$ is the pull-back of α on Σ by ϕ. Furthermore, one can check that the critical points of S which give the classical paths coincide with the critical points of the action

$$S(\phi, \gamma, \beta) \equiv \int_\Sigma \|d\phi\|^2 + \int_\Sigma \phi^*\beta$$

for any closed two form β on X. From these two remarks it follows that for a Kähler manifold (X,g) with Kähler form α, the classical σ model can equally well be described in terms of the action:

$$S(\phi,\omega) \equiv \int_\Sigma \phi^*\omega + \int_\Sigma 2\|\bar\partial\phi\|^2$$

where $\omega = \alpha + i\beta$ is now a complexified Kähler parameter, β being a closed 2-form. In fact, the actual actual parameter is the class of β in $H^2(X,\mathbb{R})/2\Pi H^2(X,\mathbb{Z})$ since an exact form contributes to the action only through a boundary term $\int_{\partial\Sigma}\phi^*\gamma$ via Stokes formula and physicists are only interested in the exponentiated action.

The action is invariant under conformal transformations of Σ. These transformations are generated by $L_n = z^n\frac{\partial}{\partial z}$.

A <u>super-symmetric σ-model</u> is classicaly described by fields $\Phi : \Sigma \to X$ where X is as before a Riemannian manifold, and Σ is a super Riemann surface, with one commuting coordinate z and one anticommuting coordinate ζ (as described in Alice Roger's talk) which minimise an action $\mathbb{C}(\Phi)$ invariant under conformal transformations in a similar way to the bosonic action S as well as one set of supersymmetries , corresponding to odd vector fields on Σ. Alvarez-Gaumé and Freedman [AF] showed that if X is a Kähler manifold there is an additionnal set of supersymmetries, so that the theory has what is known as $N = 2$ superconformal symmetry; these symmetries are maintained under quantisation provided that X is Calabi-Yau. Thus we have a mapping from the set of Calabi-Yau manifolds into the set of $N = 2$ superconformal theories; this map was a key ingredient in the discovery of mirror symmetry.

4.2. Quantisation and the Virasoro algebra

Given a classical field theory $\phi : V \to X$ (where V is a given manifold) with action $S(\phi) \equiv \int_V L(\phi,\partial_i\phi,\cdots)dvol(x)$, the space of solutions of the classical Euler equations is equipped with a Poisson structure. Noether's theorem associates functions on the space of solutions to infinitesimal symmetries of the action, in a compatible way with the Lie and Poisson brackets. Infinitesimal symmetries then yield evolution equations using the brackets $\frac{\partial}{\partial t}\phi = \{H,\phi\}$. <u>Quantising</u> the classical model means finding a representation on a Hilbert space of <u>states</u> of a set of functionals called <u>observables</u> equipped with the Poisson brackets. The quantised theory can then be reconstructed from the <u>correlation functions</u> $< f_1(x_1)\cdots f_k(x_k) >$ where $x_i, i = 1,\cdots,k$ are points on V and

$$\text{“} < f_1(x_1)\cdots f_k(x_k) > \equiv \int_{\phi\in Map(V,M)} f_1(\phi(x_1))\cdots f_k(\phi(x_k)))e^{-S(\phi)}d\phi\text{”}.$$

The integral is to be interpreted here as a formal infinite dimensional "Lebesgue integral" on path space $Map(V,M)$.

Coming back to the bosonic σ-model $\phi : \Sigma \to X$, let us notice that since the action arises in the "path integral" as an exponent, and since we have $S(\phi,\gamma,\omega) = S(\phi,\gamma) + i\int_\Sigma \phi^*\beta$, β being a form in $H^2(X,\mathbb{R})/2\pi H^2(X,\mathbb{Z})$, the correlation functions are independent of the choice of representative β modulo $2\pi\mathbb{Z}$ and are thus well-defined.

Quantisation of a conformal field theory should give a (possibly projective) representation of the Virasoro algebra (that is, the algebra generated by L and $\bar L$) and hence a theory satisfying Segal's axioms (described in [Ga]).

In a similar manner one expects quantisation of an $N = 2$ supersymmetric sigma model to lead to an $N = 2$ conformal theory, that is, essentially a representation of the super Virasoro algebra with central charge. This algebra has even generators $L_m, \bar{L}_m\, m \in \mathbb{Z}$ (as for the purely bosonic conformal group) and $J_m, \bar{J}_m\, m \in \mathbb{Z}$, together with odd generators $G_r^+, G_s^-, \bar{G}_r^-, \bar{G}_s^-, r, s \in \mathbb{Z} + \frac{1}{2}$ and an even central charge C. Full details of this algebra are given in [V] where it is also shown that there is an involution of this algebra defined by $G_r^+ \leftrightarrow G_r^-$, $\bar{G}_r^- \leftrightarrow \bar{G}_r^-$,$\bar{G}_r^+ \leftrightarrow \bar{G}_r^+$, $J_m \to -J_m, \bar{J}_m \to \bar{J}_m$. Under this involution two subrings of the representation, known as the chiral-antichiral ring R_{ca} and the chiral-chiral ring R_{cc}, are interchanged. This interchange of rings is the next step in the physicist's construction of mirror symmetry; the final step involves reversal of the steps that lead from a Calabi-Yau manifold to the (for example) chiral-chiral ring of the corresponding superconformal field theory. The chiral-antichiral ring of one superconformal theory (derived from a Calabi-Yau manifold X) is the chiral-chiral ring of a superconformal theory which can be realised by a non-linear σ- model on an different manifold X', and the idea is that these two manifolds will be a mirror pair. Now the relation with geometry is given by the description of the (anti)chiral rings in terms of the Dolbeault cohomology of the target manifold of the σ-model. The relation with mirror symmetry comes from the fact that the rings R_{ca} and R_{cc} have bigradations $R_{ca}^{p,q}$ and $R_{cc}^{p,q}$ corresponding to eigenvalues of J_0 and $\bar{J}_0$, with $R_{c,c}^{p,q} \cong H^q(\Lambda^p T_X)$ and $R_{c,a}^{-p,q} \cong H^q(\Lambda^p \Omega_X)$.

Witten suggests a more geometric interpretation, whereby mirror symmetry would exchange two supersymmetric quantum field models, a model A built from X and a model B built from its mirror X' thus exchanging special correlation functions called Yukawa couplings computed for each of these models, which we shall denote by Y for the model A and Y' for the model B. As we shall see below, the (normalised) Yukawa coupling Y is an n-symmetric form on $H^1(T_X)$ and Y' is an n-symmetric form on $H^1(\Omega_{X'})$.

5. Mirror Symmetry and Mathematics

The construction of the mirror pair of a Calabi-Yau manifold using superconformal field theory is both indirect and (in places) tenuous. It is natural to ask whether there is not some more direct and rigorous way of obtaining the mirror pair. At present no such method is known, but considerable progress has been made towards this goal.

5.1. Mathematical evidence for mirror symmetry

Properties of Yukawa couplings give a hint towards mirror symmetry.

Let X be a Calabi-Yau manifold of dimension 3 such that $h^1(X) = 0$. For a given $\kappa \in H^{3,0}(X)$ where $H^{3,0}(X)$ enters in the Hodge decomposition $H^3(X,\mathbb{C}) = H^{3,0}(X) \oplus H^{1,2}(X) \oplus H^{2,1}(X) \oplus H^{0,3}(X) =$, one defines a Yukawa coupling Y_κ on X as a symmetric three form on $H^1(T_X)$:

$$Y_\kappa(u_1, u_2, u_3) \equiv < \kappa^2, u_1 \cdot u_2 \cdot u_3 >$$

where $< \cdot, \cdot >$ corresponds to Serre duality (see e.g [GH] chap.1), $u_1 \cdot u_2 \cdot u_3 \in H^3(\Lambda^3 T_X)$ where $u_1 \cdot u_2 \cdot u_3$ denotes the cup-product [GH] of u_1, u_2, u_3.

These Yukawa couplings have a remarkable property, namely that they arise as "derivatives" of a potential for a natural choice of coordinates on M_X. In other words, there is a special choice for the parameter κ and there are special coordinates $z_1, \cdots, z_N$, $N = \dim H^1(T_X)$ on M_X (depending on a choice of symplectic basis on $H^3(X, \mathbb{Z})$ for the intersection form) and a function $F(z_1, \cdots, z_N)$ such that (see [V] Proposition 3.3)

$$Y_\kappa(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}) = \frac{\partial^3}{\partial z_i \partial z_j \partial z_k} F$$

This construction also yields a flat structure (coordinates defined up to affine transformation) on M_X which depends on the symplectic basis chosen on $H^3(X, \mathbb{C})$. (see Lemma 3.2 in [V]).

5.2. Predictions of mirror symmetry

Confronting these results with the results predicted by mirror symmetry gives evidence for mirror symmetry.

Let X be as before a three dimensional Calabi-Yau manifold with $h^1(X) = 0$. If mirror symmetry exists, there is a local identification between deformations of complex structures and deformations of the Kähler parameter on its mirror X'. Since there is a canonical flat structure on the Kähler deformations of X', mirror symmetry predicts the existence of a flat structure on the space of deformations of the complex structures on X.

Let M be the mirror map; $Y'_\kappa \equiv M^*(Y_\kappa)$ gives a cubic form on $H^1(\Omega_{X'})$ which depends only on the Kähler parameter. One beautiful prediction due to the physicists (cf Witten) is the description of Y' in terms of cubic derivatives of the Gromov-Witten potential G- the definition of which involves Gromov-Witten invariants of Y'_κ (see [A], [LY], [V]). Hence with the privileged choice of coordinates on either side as described above, the mirror map M has to be affine linear and one expects that $M^*F = G$, F being the potential defined above for Y_κ (in fact up to a quadratic function of the flat coordinates).

This identification predicts the number of rational curves in a quintic of any degree. Predictions have been checked up to degree 4 (see [ES], [LY]).

These are only a few hints for mirror symmetry. There have been many other investigations made in that direction, such as Batyrev's combinatorial construction of a beautiful series of examples, for which we refer the reader to [V].

References

These are only a few references around the topic of mirror symetry. We refer the reader to [V] for further references.

[A] M. Audin, *Cohomologie quantique*, Bourbaki vol 1995-96, Exposé 806 , Nov 1995

[AF] L.Alvarez-Gaumé, D. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric σ-model*, Comm. Math. Phys. 80 p.443-451 (1981)

[D] R. Dijkgraaf, *Mirror symmetry and elliptic curves,* in The Moduli Space of Curves (R.Dijkgraaf, C.Faber, G. Van der Geer Eds.), Birkhäuser, 1995

[Ga] K.Gawedski, *Conformal field theory*, Séminaire Bourbaki, exp. 704, 1988-89

[G] D.Gepner, *exactly solvable string compactifications on manifolds of $SU(n)$-holonomy*, Physics Letters B, Vol. 199, n 3, p.380-388

[GH] P.Griffiths, J. Harris, *Principles of algebraic geometry*, Wiley-Interscience Publication, 1976

[GP] B.R Greene, M.R Plesser *An introduction to Mirror Manifolds,* in "Essays on Mirror Manifolds" , Ed. S.T. Yau, International Press

[ES] G.Ellingsrud, S.A. Stromme, *The number of twisted cubics on the general quintic threefold*, in "Essays on Mirror Manifolds" , Ed. S.T. Yau, International Press

[LY] B.H.Lian, S.T. Yau Mirror Symmetry, Rational curves on algebraic manifolds and hypergeometric series, Proceedings of the XIth International Congress of Mathematical Physics, Ed. D.Iaglnitzer (1995)

[M] D.Morrison, *Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians*, Journ. Amer. Math. Soc. 6 (1993), p.223-247

[V] C. Voisin, *Symétrie Miroir*, Panoramas et synthèses, Société Mathématique de France, 1996 (to appear)

[W] E. Witten, *Mirror manifolds and topological field theory* in "Essays on Mirror Manifolds", Ed. S.T. Yau, International Press p. 120-158

Proc. of the 7^{th} EWM meeting, Madrid, 1995

Modelling the randomness in physics

Flora Koukiou

Université de Cergy-Pontoise, France
koukiou@u-cergy.fr

Randomness is one of the paradigms of modern physics. During the last years a wide variety of models of disordered statistical mechanics have been introduced and studied. By disordered system we mean a system with quenched (frozen-in) randomness which vary from sample to sample. For modelling randomness one can either introduce random pertubations in an ordered system, or consider random interactions between the different components of the system. In the case of perturbations, an important question to address is about the stability of phase transitions under random perturbations. The case of random interactions (spin glasses) is much more difficult and still heavily debated. We can however use a mean field theory to study the behaviour of the system by neglecting the effects of fluctuations. In many cases fluctuations are irrelevant: systems in sufficiently many spatial dimensions or with long-range interactions (each component interacts with each other component). In mathematical physics, mean field models are usually provided by systems defined on complete graphs or trees.

In the following, we present a simple application of the theory of branching random walks to the mean-field theory of a random systems defined on regular trees. In a tree of coordination number $d \geq 2$, one can study mean field models of spin glasses or directed polymers using the theory of branching random walks.

We can define a simple model on a d-tree (*i.e.* each vertex has d edges; for instance the dyadic tree is a 2-tree) as following:

Let $\mathcal{T}_n$ be a finite d-tree. If v is a vertex, we denote by $|v|$ the number of edges (or steps) of the path going from the root to v. We have thus d^n paths p of $|v| = n$ steps. On this d-tree we can do simple random walks starting at the root and choose any of the d edges coming out independently. Let $w(\beta)$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$, depending on a real variable β called temperature. We assume moreover that the variable $w(\beta)$ has moments of all orders and that $w^{-1}(\beta)$ is almost surely bounded. To each edge $b \in \mathcal{T}_n$ we assign the random energy $w_b(\beta)$ having common distribution with $w(\beta)$. We have thus a family of independent identically distributed random variables $\{w_b(\beta),\ b \in \mathcal{T}_n\}$ indexed by the edges of the tree. In models of statistical physics, the variables $\{w_b(\beta)\}$ are associated with the Boltzmann factors. In the case of random interactions between the components of a system, each path p from the root to the vertex v, corresponds to a spin configuration whose Gibbs weight is given by the sum of energies over the $|v|$ edges. In the case of directed polymers the energy of each walk is given by the product of the energies of the visited edges.

[0]Work partially supported by the EU grant CHRX-CT93-0411

Having this in mind, we can define on $\mathcal{T}_n$, the partition function of the models by

$$Z_n(\beta) = \sum_p \prod_{b \in p} w_b(\beta),$$

the specific free energy by

$$F_n(\beta) = n^{-1} \log Z_n(\beta).$$

Moreover, the Gibbs distributions can be defined by the following random measures

$$\nu_{n,\beta}(\cdot) = \frac{\prod_{b\in p} w_b(\beta)}{Z_n(\beta)}.$$

The main object of interest is the behaviour of the previous quantities at the macroscopic, or thermodynamic, $(n \to \infty)$ limit as function of β. In order to study this behaviour we can express the previous defined thermodynamic quantities as random measures. Remarking that the d-adic partition of the unit interval corresponds to the d-tree, we can define and study random measures on the unit interval which are related to $Z_n(\beta)$ and $\nu_{n,\beta}$.

In many cases, there exists a value of β, called critical temperature, β_c, such that for $\beta < \beta_c$:

- $Z_n(\beta)$ goes to a non zero limit as $n \to \infty$;
- the limit
$$F_\infty(\beta) = \lim_{n\to\infty} n^{-1} \log Z_n(\beta)$$
exists almost surely and is given by $n^{-1} \log EZ_n(\beta)$ ($E(\cdot)$ means the expectation). This result gives the existence and the so-called self-averaging property of the free energy, *i.e.*, the coincidence of the annealed
$$\lim_{n\to\infty} \frac{1}{n} \log EZ_n(\beta)$$
and quenched limits
$$\lim_{n\to\infty} \frac{1}{n} E \log Z_n(\beta);$$
- $\nu(n, \beta)$ has a unique (weak) limit $\nu(\cdot)$, as $n \to \infty$.

.

On the other hand, for $\beta \geq \beta_c$ we have

- $Z_n(\beta)$ goes to zero as $n \to \infty$;
- the limit $F(\beta)_\infty = \lim_{n\to\infty} n^{-1} \log Z_n(\beta)$, exists almost surely and it is a non random quantity. Using large deviation techniques we can explicitely calculate this limit;
- in general, there are many limits of $\nu_{n,\beta}(\cdot)$.

The phenomenon of phase transition is expressed by the previous setting. The region of $\beta < \beta_c$ is called the high temperature region and $\beta \geq \beta_c$ defines the low temperature region. This simple model provides a general framework for all mean field models studied by physicists and gives some insight to our understanding of phases transitions in random systems. The interest reader can found the detailed definitions and proofs in "The mean field theory of directed polymers in random media and spin glass models ", Rev. Math. Phys. **7**, 183–192, (1995).

Proc. of the 7^{th} EWM meeting, Madrid, 1995

Super moduli space

Alice Rogers

King's College, London, England
f.a.rogers@kcl.ac.uk

1. Introduction

Moduli spaces have been one of the themes of this meeting; in this paper my aim is to show how moduli spaces may be used in the functional integration approach to the quantisation of systems with symmetry, with particular reference to the super moduli space of super Riemann surfaces and the spinning string. At the outset it should be made clear that the term 'super' implies an extension of some standard object to include anticommuting elements in some sense, the term deriving its name from the notion of supersymmetry in physics.

The Feynman path integral was initially developed as a method for determining the time evolution of a quantum system; for some systems the path integral formulation may be derived rigorously from canonical quantisation. A generic feature of this approach, even when extended either rigorously or heuristically to quantum field theory (in which case path integrals become functional integrals) is the emergence of the term $\exp(-iS)$ (where S is the action of the theory), so that covariance, absent in the canonical approach, is restored in the path integral. This suggests that the path integral may be a more fundamental starting point than the canonical approach; and such integrals are widely used to define the quantisation of a theory when there is no direct derivation of the functional integral by canonical methods. There remain many features of functional integrals which are not well understood, but formal manipulations have led to remarkable insights in both mathematics and physics, so that the pursuit of a proper understanding of these integrals seems highly desirable.

In this paper the basic idea of the Feynman path integral in quantum mechanics is described, and a very heuristic description given of the extension of these idea to functional integrals in quantum field theory, and the modification needed when a system has a gauge symmetry. In Section 9 these ideas are applied to the bosonic string, and it is shown that the functional integral reduces to an integral over the moduli spaces of Riemann surfaces. In the final section we see that the spinning string leads to an integral over a super moduli space.

2. The Feynman path integral in quantum mechanics

In quantum mechanics a key equation is the Schrödinger equation

$$\frac{\partial f}{\partial t} = -iHf \tag{1}$$

which determines the time evolution of the wave function f of a system whose Hamiltonian is H. (The Hamiltonian is a sef-adjoint operator on the Hilbert space $\mathcal{H}$ of wave functions.)

For the case of a single particle of unit mass moving in one dimension under a field of force derived from a potential $V(x)$, wave functions are square integrable functions on the real line (with differentiable dependence on time as well) and the Hamiltonian takes the form

$$H = H_0 - V(x) \tag{2}$$

where $H_0 = -\frac{1}{2}\frac{d^2}{dx^2}$. The system is solved if the operator $\exp(-iHt)$ is known, since this leads to solutions of the Schrödinger equation (1). It is sufficient to determine the kernel of this operator, that is, the function $\exp(-iHt)(q_I, q_F)$ which satisfies

$$\exp(-iHt)f(q_F) = \int dq\, \exp(-iHt)(q, q_F) f(q). \tag{3}$$

(Here we are being vague about analytic details, and cheerfully assuming kernels exist; this of course depends on the nature of V; all the analytic questions become much more tractable if one considers $\exp(-Ht)$. A good entry point to the literature on these matters is [1].) Since $\exp(-iHt)\exp(-iHs) = \exp(-iH(s+t))$, the kernels of these three operators are related by the involution

$$\exp(-iH(s+t))(q_I, q_F) = \int dq\, \exp(-iHs)(q_I, q)\exp(-iHt)(q, q_F). \tag{4}$$

The trick required to derive the path integral expression for the kernel of $\exp(-iHt)$ is to use the fact that

$$\exp(-iHt) = \left(\exp(-\frac{iHt}{N})\right)^N \approx \left(\exp(-\frac{iV(q)t}{N})\right)^N \left(\exp(-\frac{iH_0t}{N})\right)^N \tag{5}$$

for large N. Then the involution (4) repeated $N-1$ times gives

$$\exp(-iHt)(q_I, q_F) = \int \prod_{i=1}^{N-1} dq_i \prod_{i=1}^{N} \exp(-iH\Delta t)(q_{i-1}, q_i) \tag{6}$$

where $\Delta t = \frac{t}{N}$, $q_0 = q_I$ and $q_N = q_F$, and thus, if N is sufficiently large,

$$\begin{aligned}
\exp(-iHt)(q_I, q_F) &\approx \int \prod_{i=1}^{N-1} dq_i \prod_{i=1}^{N} \exp(-iV(q_i)\Delta t)(\exp(-iH_0\Delta t)(q_{i-1}, q_i)) \\
&\propto \int \prod_{i=1}^{N-1} dq_i \prod_{i=1}^{N} \exp(-iV(q_i)\Delta t)\exp\left(\frac{i(q_i - q_{i-1})^2}{2\Delta t}\right) \\
&= \int \prod_{i=1}^{N-1} dq_i \exp\left[i\sum_{i=1}^{N}\left(-V(q_i)\Delta t + \left(\frac{q_i - q_{i-1}}{2\Delta t}\right)^2\right)\Delta t\right] \\
&= \int \mathcal{D}q \exp\left(i\int_0^t (-V(q(s)) + \tfrac{1}{2}(\dot{q}(s))^2 ds\right)
\end{aligned} \tag{7}$$

where $\mathcal{D}q$ denotes that the integral is to be taken over all paths $q(s)$ satisfying $q(0) = q_I$ and $q(t) = q_F$. The expression in the final line is essentially defined by the line above. (The argument presented here roughly follows the ground-breaking work of Feynman and Hibbs [2].)

The final expression has the classic form

$$\int \mathcal{D}q \exp -iS(q(\cdot))$$

where $S = \int_0^t \left(\frac{1}{2}(\dot{q}(s))^2 - V(q(s)\right) ds$ is the action of the system. As mentioned before, physicists proceed to quantise a vast range of systems by investigating this integral for the appropriate action. For instance, in quantum field theory, where instead of a single configuration variable $q(t)$ there is one for each point $\mathbf{x}$ in space, the integrals become sums over all functions (or fields) $\phi(\mathbf{x}, t)$ on spacetime, and the path integral is written

$$\int \mathcal{D}\phi \exp -iS(\phi(\cdot, \cdot)) \tag{8}$$

where $S(\phi(\cdot, \cdot))$ denotes the action of the system.

When the system has a symmetry, that is, some group G acts on the space of fields in such a way that the action $S(\phi(\cdot, \cdot)$ is invariant, the functional integral is taken of the space of fields modulo the action of the group G. This space is in general smaller than the full space of functions, but may be more complicated. However in the case of some very highly symmetric theories, particularly string theories and topological field theories, the space is finite-dimensional, and has an interesting geometrical interpretation.

3. The moduli space for string path integrals

The first example I shall consider of a theory where the function space is reduced to a finite-dimensional moduli space is closed bosonic string theory, following the approach of Polyakov [3], which is explained together with many mathematical developments by Bost [4]. In this case the classical action is

$$S(g(\cdot), X(\cdot)) = \int_\Sigma d^2x \sqrt{\det g_{ij}(x)} g^{ij}(x) \partial_i X^a \partial_j X^b \eta_{ab} \tag{9}$$

where Σ is a 2-dimensional surface, g_{ij} are the components of a Riemannian metric g on Σ, X is a mapping of Σ into $\mathbb{R}^d$ (a consistent quantum theory being obtained when d, the dimension of the space-time $\mathbb{R}^d$ in which the string moves, is equal to 26), and η_{ab} is the Minkowski metric in $\mathbb{R}^d$. To quantise the theory the functional integral

$$\int \mathcal{D}X \mathcal{D}g \exp(-iS(X(\cdot), g(\cdot))$$

must be evaluated over the space of all fields X, g, modulo the symmetries of the theory. It is sufficient to consider only Σ which are compact surfaces without boundary, summing the results for each possible genus. The action of the theory is symmetric both under the action of the diffeomorphism group of the surface Σ and under conformal transformations $g \mapsto e^{\phi(x)} g$. It is possible to carry out the X integration explicitly, since this integral is a Gaussian, so that it remains to integrate a function of the metric g over the space of all possible g modulo the symmetries. It turns out that this space is simply the moduli space of all possible complex structures on Σ. To see this, suppose that g has components g_{ij} with respect to local coordinates $x^i, i = 1, 2$ on Σ; then it is always possible to choose a diffeomorphism (and hence new local coordinates $y^i, i = 1, 2$) on Σ such that the new components have the form

$g_{ij} = e^{\phi(x)}\delta_{ij}$. Suppose that $\hat{y}^1, \hat{y}^2$ are some other coordinates where the components of g are also diagonal, and that z and $\hat{z}$ are defined as $z = y^1 + iy^2$, $\hat{z} = \hat{y}^1 + i\hat{y}^2$. Then (by standard arguments in complex function theory) z is an analytic function of $\hat{z}$. Thus each conformal and diffeomorphism class of metrics determines a complex structure on Σ, and the space over which the functional integral for the closed bosonic string must be carried out is the moduli space of complex structures on the surface Σ. This space has been much studied, and is a finite-dimensional manifold with singularities. In the next section it will be seen that the analogous space for the spinning string is a super moduli space.

4. Super moduli space and the spinning string

To incorporate fermions (that is, particles with half integral spin) into string theory, the spinning string is introduced. The geometric theory of the spinning string can be formulated using anticommuting variables, as first described by Howe [5]. The ingredients are complicated but have become standard in supergravity theory. They involve the notion of supermanifold, which can be defined in a bewildering number of different ways; however a quite naive approach is sufficient for this lecture. The action of the spinning string is

$$S(E_M^A) = \tfrac{1}{4}\int_\Sigma d^2x\, d^2\theta \text{ superdet } (E_M^A) D_\alpha V D^\alpha V \tag{10}$$

where x^1, x^2 are even commuting coordinates and θ^1, θ^2 are odd anticommuting coordinates on the $(2,2)$-dimensional supermanifold Σ, E_M^A is a generalisation of a metric on Σ (described below), V is a function on Σ and D_α is a differential operator whose precise definition is not important here.

The object E_M^A is known as a vielbein and generalises the zweibein version of a metric: on a standard 2-dimensional manifold a metric is a symmetric non-degenerate quadratic form on the tangent space; thus there exist orthonormal bases $e_a, a = 1, 2$ of the tangent space; such a basis is not unique, there is an SO(2) bundle of orthonormal frames. This bundle contains all the data of the metric. With respect to local coordinates the dual basis e^a of one-forms may be expanded as

$$e^a = e_m^a\, dx^m. \tag{11}$$

A basis e^1, e^2 is known as a zweibein.

Returning to the supermanifold, we consider a reduction of the frame bundle (which is in fact a super group bundle) to an SO(2) bundle, with action on the preferred frames taking the form

$$\begin{pmatrix} E_a \\ E_\alpha \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} E_a \\ E_\alpha \end{pmatrix} \tag{12}$$

where E_a and E_α are respectively the two odd and two even elements of the preferred basis of the tangent space, and R is an element of SO(2). The allowed vielbein are constrained so that the bundle omits a connection with some components of the torsion taking a prescribed form. (This is a physical requirement.)

Expanding the dual basis E^a, E^α of the cotangent space in terms of the coordinate basis x^m, θ^μ we have 16 components of the vielbein forming an invertible matrix

$$\begin{pmatrix} E_m^a & E_m^\alpha \\ E_\mu^a & E_\mu^\alpha \end{pmatrix}.$$

The superdeterminant of a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of this nature is defined to be $\det(A - BD^{-1}C)(\det D)^{-1}$. (It is a non-trivial property of the superdeterminant that it obeys the multiplicative rule.)

The final geometrical ingredient needed is a notion of integration; this is defined by

$$\int d^2x\, d^2\theta\, (f(x) + f_1(x)\theta^1 + f_2(x)\theta^2 + f_{12}(x)\theta^1\theta^2) = \int d^2x\, f_{12}(x). \tag{13}$$

The super conformal geometry emerges from the symmetries of the theory; these are superdiffeomorphisms of Σ, together with so-called super Weyl transformations [5]

$$\begin{pmatrix} E_m^a & E_m^\alpha \\ E_\mu^a & E_\mu^\alpha \end{pmatrix} \mapsto \begin{pmatrix} \Lambda E_m^a & \Lambda^{\frac{1}{2}} E_m^\alpha - \frac{1}{2}\Lambda^{-\frac{1}{2}} E_m^a \gamma_a^{\alpha\beta} D_\beta \Lambda \\ \Lambda E_\mu^a & \Lambda^{\frac{1}{2}} E_\mu^\alpha - \frac{1}{2}\Lambda^{-\frac{1}{2}} E_\mu^a \gamma_a^{\alpha\beta} D_\beta \Lambda \end{pmatrix}.$$

(Here $\gamma_a^{\alpha\beta}$ are the Dirac γ-matrices which represent the 2-dimensional Clifford algebra associated with spin representations of SO(2).)

These are the simplest transformations which preserve the constraints on the connection without restricting the parameter $\Lambda(x, \theta)$, except by requiring that it is invertible everywhere. The superdifeomorphism symmetry is used to choose a coordinate system where the vielbein is 'super-Weyl flat', that is, obtainable from the matrix

$$\begin{pmatrix} \delta_m^a & 0 \\ i\theta^\lambda \gamma_{\lambda\alpha}^a & \delta_\mu^\alpha \end{pmatrix} \tag{14}$$

by a super Weyl transformation. If one now chooses complex coordinates z, ζ with $z = x^1 + ix^2$ and $\zeta = \theta^1 + i\theta^2$, then changes of coordinates $(z, \zeta) \mapsto (\tilde{z}, \tilde{\zeta})$ which preserve super Weyl flatness of the vielbein are what is known as superconformal: $\tilde{z}$ and $\tilde{\zeta}$ are both superanalytic functions of z and ζ [6], (so that

$$\begin{aligned} \tilde{z} &= f(z) + \zeta\phi(z) \\ \tilde{\zeta} &= \psi(z) + \zeta g(z) \end{aligned} \tag{15}$$

with all four functions $f(z), \phi(z), \psi(z), g(z)$ analytic) and also the differential operator $D = \frac{\partial}{\partial\zeta} + \zeta\frac{\partial}{\partial z}$ transforms mutiplicatively with

$$D = (D\tilde{\zeta})\tilde{D}. \tag{16}$$

The nature of the transition functions from z, ζ to $\tilde{z}, \tilde{\zeta}$ demonstrates that the surface Σ has the structure of a particular kind of $(1,1)$-dimensional complex supermanifold known as a *super Riemann surface.*(The extra condition (16) means that the change of coordinates takes the form

$$\begin{aligned} \tilde{z} &= f(z) + \zeta\psi(z) \\ \tilde{\zeta} &= \psi(z) + \zeta\sqrt{f'(z) + \psi(z)\psi'(z)}. \end{aligned} \tag{17}$$

For future reference, it may be observed that by setting $\psi(z)$ to zero and ignoring ζ a conventional Riemann surface, known as the body of the super Riemann surface, is obtained.

A super Riemann surface with simply connected body is said to be simply connected, and similarly for any other topological attribute.)

From the analysis above we see that the functional integral for the spinning string (equation (10)) reduces to an integral over the super moduli space of all possible super Riemann surfaces, and so one is led to a study of the nature of this space. Following Crane and Rabin [7], the following picture emerges: three simply-connected super Riemann surfaces can be found which are super extensions of the three simply connected Riemann surfaces, and then a uniformisation theorem established to show that any other compact super Riemann surface without boundary is a quotient of one of these three simply connected super Riemann surfaces by a discrete subgroup of the superconformal automorphism group. The starting point is the observation that corresponding to any Riemann surface with spin structure a super Riemann surface can be constructed with transition functions

$$\begin{aligned} \tilde{z} &= f(z) \\ \tilde{\zeta} &= \zeta\sqrt{f'(z)}, \end{aligned} \tag{18}$$

where f is the transition function on the Riemann surface and the sign of $\sqrt{f'(z)}$ is determined by the spin structure. Since each of the three simply connected Riemann surfaces (the complex plane $\mathbb{C}$, the complex sphere $\mathbb{C}_*$ and the upper half plane U) has a unique spin structure, they each possess exactly one super extension of this nature, denoted $S\mathbb{C}$, $S\mathbb{C}_*$ and SU respectively. Crane and Rabin [7] use cohomological arguments to show that these are the only simply connected super Riemann surfaces, and hence establish their uniformisation theorem. The moduli space corresponding to each genus and spin structure can then be investigated. At genus 0, the only possible super Riemann surface is $S\mathbb{C}_*$, while at genus 1 various toroidal compactifications are possible; for even spin structures on the body, these compactifications are parametrised by a single even parameter for even spin structures, so that the moduli space is $(1,0)$-dimensional, while for the odd spin structure the moduli space has dimension $(1,1)$. Super Riemann surfaces of higher genus are obtainable as quotients of SU by a discrete subgroup Γ of the group of conformal automorphisms of the upper half plane which consists of transformations of the standard form (17) with [7]

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ \psi(z) &= \frac{\gamma z+\delta}{cz+d} \end{aligned} \tag{19}$$

where a, b, c and d are all real and even and satisfy $ad - bc = 1$ while γ and δ are real and odd. Now by the usual arguments, Γ must be isomorphic to the fundamental group of a surface of genus g, where g is the genus of the body of the super Riemann surface SU/Γ, and this group has $2g$ generators and one relation; thus, allowing for a freedom of overall conjugation, there will be $3(2g-2)$ even parameters to be chosen and $2(2g-2)$ odd parameters to be chosen to determine a particular Γ. Thus supermoduli space is (at least to the extent that the corresponding result is true for standard Riemann surfaces) a finite-dimensional supermanifold, and we see that the geometrical formulation of the spinning string given by Howe [5] leads eventually to the reduction of the functional integral for the spinning string to a finite-dimensional integral. A further account of the geometry of super Riemann surfaces and its application to string theory may be found, for example, in [8, 9].

References

[1] B. Simon. *Functional Integration and Quantum mechanics.* Academic Press, 1979.

[2] R.P. Feynman and A. Hibbs. *Quantum Mechanics and Path Integrals.* McGraw-Hill, 1965.

[3] A.M. Polyakov. Quantum geometry of bosonic strings. *Pysics Letters*, 103B:207–210, 1981.

[4] J.-B. Bost. Fibrés déterminants régularisés et mesures sur les espaces de modules des courbes complexes. *Séminaire Bourbaki*, 676:1–36, 1986-7.

[5] P.S. Howe. Super Weyl transformations in two dimensions. *Journal of Physics A*, 12:393–402, 1979.

[6] D. Friedan. Unified string theories. In M. Green and D. Gross, editors, *Proceedings of the Workshop on Unified String Theories*, 1986.

[7] L. Crane and J.M. Rabin. Super Riemann surfaces: uniformization and Teichmüller theory. *Communications in Mathematical Physics*, 113:601–623, 1988.

[8] A.A. Rosly, A.S. Schwarz, and A.A. Voronov. Geometry of superconformal manifolds. *Communications in Mathematical Physics*, 119:129–152, 1988.

[9] A.A. Rosly, A.S. Schwarz, and A.A. Voronov. Superconformal geometry and string theory. *Communications in Mathematical Physics*, 120:437–450, 1989.

Proc. of the 7^{th} EWM meeting, Madrid, 1995

Universal measuring coalgebras: The points of the matter

Marjorie Batchelor

Cambridge University, England

1. Introduction

One of the challenges modern theoretical physics has put to classical mathematics is the necessity of finding a satisfactory concept of manifold or point set which will accomodate the increasingly varied requirements. Supersymmetry necessitated the introduction of supermanifolds. String theories and field theories in general require the infinite dimensional manifolds of smooth maps. The geometric interpretation of non-commutative geometry is still puzzling.

The purpose of this paper is to introduce the universal measuring coaglebra and to suggest that many of these challenges can be met by this construction. It provides a solution to problems of differential topology in supermanifolds and the spaces of maps. There is evidence that it may give a satisfactory interpretation of quantum groups and similar objects. Moreover as its construction is a universal one requiring only that the input objects are algebras it ought to have applications to other categories which depend on a notion of function algebra. Finally, because of the strong built in finiteness properties possessed by coalgebras, this construction in some sense encodes all finite dimensional information.

I will introduce measuring coalgebras, describe how they function as surrogate point sets, and how they can be used to get jet bundle information, in both the classical case of finite dimensional manifolds, and the case of graded manifolds. A further slight generalization allows one to get hold of the jet bundle of the manifold of smooth maps between (ordinary or graded) manifolds. The final example I will give is the simplest one which demonstrates the possibility of representing quantum groups as “transformation groups”.

2. Definitions of Coalgebras

Motivation Every point set S comes equipped with a diagonal map $S \to S \times S$ taking a point s to the pair (s, s). This map is used implicitly in the usual definition of multiplication of functions on S: if f, g are two (say real valued) functions on S the product fg is defined by

$$fg(s) = f(s)g(s).$$

This definition depends evidently on multiplication in $\mathbb{R}$, but equally on the diagonal map $s \to (s, s)$.

The view I take is that the properties which define the notion of point are the following:

1. Points take values on functions; that is, they are linear functionals on a (given) algebra F of functions.

2. Points are precisely those functionals s which satisfy the product rule

$$fg(s) = f(s)g(s).$$

determined by multiplication in $\mathbb{R}$ and the usual diagonal map.

This is a viewpoint which generalizes easily, coalgebras providing sets equipped with generalizations of the diagonal map, and measuring maps providing the idea of a product rule.

Definition A coalgebra is a linear space C together with a comultiplication

$$\Delta : C \longrightarrow C \otimes C$$

and a counit

$$\varepsilon : C \longrightarrow \mathbb{R}$$

satisfying the following identities.

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle\Delta\otimes 1} \\ C \otimes C & \xrightarrow{1\otimes\Delta} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle\varepsilon\otimes 1} \\ C \otimes C & \xrightarrow{1\otimes\varepsilon} & C \end{array}$$

Examples

i $C = \mathbb{R}m$, $\Delta m = m \otimes m$, $\varepsilon m = 1$. This is the coalgebra with point-like behaviour.

ii $C = \mathbb{R}m + \mathbb{R}t$, $\Delta m = m \otimes m$, $\varepsilon m = 1$, $\Delta t = t \otimes m + m \otimes t$, $\varepsilon t = 0$. Observe that comultiplication of t resembles the product rule for derivations. Elements with that comultiplication are called primitive.

iii $C = \mathbb{R}k + \mathbb{R}h + \mathbb{R}E$. The elements h, k are pointlike (as m in example **i**), and $\Delta E = k \otimes E + E \otimes h$, $\varepsilon E = 0$. This coalgebra is the one which arises in quantum groups.

Notation Following Sweedler, the comultiplication Δc will be written

$$\Delta c = \sum_{(c)} c_{(1)} \otimes c_{(2)}.$$

The coassociativity diagram allows us to write

$$(\Delta \otimes 1)\Delta c = \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = (1 \otimes \Delta)\Delta c.$$

The counit diagram above states that

$$c = \sum_{(c)} \varepsilon(c_{(1)})c_{(2)} = \sum_{(c)} c_{(1)}\varepsilon(c_{(2)}).$$

Pleasing property of coalgebras The defining property of tensor products makes it easy to construct maps from tensor products not to them. The consequence of requiring the comultiplication map to go "the wrong way" is that coalgebras have a very strong built in finiteness property: *every element in a coalgebra C lives in a finite dimensional subcoalgebra of C.* This has a number of desirable consequences on which we can capitalize.

3. Measuring coalgebras

This is the concept of maps between algebras which satisfy a product rule.

Definition Let A, B be algebras (over $\mathbb{R}$ for example) and let C be a coalgebra. A linear map

$$\Psi : C \longrightarrow \mathrm{Hom}_{\mathbb{R}}(A, B)$$

is said to *measure* if $\Psi(c)(a_1 a_2) = \sum_{(c)} \Psi(c_{(1)})(a_1)\Psi(c_{(2)})(a_2)$ and $\Psi(c)(1_A) = \varepsilon(c)1_B$.

Examples

i If C is the coalgebra of example **i** above, then a linear map $\Psi : C \to \mathrm{Hom}_{\mathbb{R}}(A, B)$ measures if and only if $\Psi(m)$ is an algebra homomorphism.

ii If C is the coalgebra of example **ii** above, a linear map $\Psi : C \to \mathrm{Hom}_{\mathbb{R}}(A, B)$ measures if and only if $\Psi(m)$ is an algebra homomorphism, and $\Psi(t)$ is a derivation with respect to $\Psi(m)$.

Not only are there familiar examples, then, of measuring coalgebras, but also there exists a "maximal" measuring coalgebra for a given pair (A, B) of algebras.

Definition A measuring coalgebra $\pi : P \to \mathrm{Hom}(A, B)$ is a *universal measuring* coalgebra if given any other measuring coalgebra $\Psi : C \to \mathrm{Hom}(A, B)$ there is a unique map (of coalgebras) ρ which makes the following diagram commute.

$$\begin{tikzcd} P \arrow[r, "\pi"] & \mathrm{Hom}(A, B) \\ C \arrow[u, "\rho"] & \end{tikzcd}$$

The existence of universal measuring coalgebras depends on the pleasing property of coalgebras. Specifically, that finiteness property guarantees that in the category of coalgebras, coproducts and colimits exist. This is enough to construct a suitable measuring coalgebra P with the required properties. Such objects are unique by general categorical principles.

Notation The universal measuring algebra for the pair (A, B) will be denoted $P(A, B)$.

4. Applications to manifolds and graded manifolds

The starting point for this discussion has been that a point set S is a subset of linear functionals on a (given) function algebra F. The idea then is to replace S by $P(F, \mathbb{R})$. As a test piece we can calculate $P(C^\infty(M), \mathbb{R})$ where M is a smooth manifold and $C^\infty(M)$ is the algebra of smooth functions on M.

We know some elements of $P(C^\infty(M), \mathbb{R})$ already. If m is a point in M, then the assignment $f \to f(m)$ is an algebra homomorphism. Thus by example **i** above, $\mathbb{R}m \to \mathrm{Hom}(C^\infty(M), \mathbb{R})$ measures, and by the universal property of $P(C^\infty(M), \mathbb{R})$, $\mathbb{R}m$ is included in $P(C^\infty(M), \mathbb{R})$. Denote the image of $\mathbb{R}m$ under this inclusion by T^0_m.

Similarly, if γ is a tangent vector to M at m, so that γ is in the tangent space $T_m M$, the assignment $t \to \gamma$ determines a measuring map from $\mathbb{R}m + \mathbb{R}t \to \mathrm{Hom}(C^\infty(M), \mathbb{R})$. In this way $\mathbb{R}_m + T_m M$ can be considered as a measuring coalgebra for $(C^\infty(M), \mathbb{R})$. Denote the image of this measuring coalgebra in $P(C^\infty(M), \mathbb{R})$ by T^1_m.

Higher order tangent spaces can be treated similarly and give an increasing union of subcoalgebras T^k_m of $P(C^\infty(M), \mathbb{R})$ as in Figure 1.

Figure 1:

Result

$$P(C^\infty(M), \mathbb{R}) = \sum_{m \in M} \bigcup_k T^k_m.$$

Each subcoalgebra T^k_m is dual to the fibre of the k^{th} jet bundle of M at m The sum over points in m is direct.

Thus in the familiar case of smooth manifolds, this entirely algebraic construction recovers all the jet bundle information of the manifold M. It is therefore an attractive candidate to use to recover jet bundle information when conventional manifold techniques fail to apply. A first application is to graded manifolds.

Graded manifolds are manifolds whose function algebra includes anticommuting elements, that is, functions f, g such that $fg = -gf$. They are defined as follows.

Definition A *graded manifold* is a pair (M, A) where M is an ordinary smooth manifold, and A is a sheaf of graded commutative algebras such that there is an atlas of open charts on M such that over such an open set the sheaf A is isomorphic to the sheaf $C^\infty(.) \otimes \Lambda\mathbb{R}^n$. The basis $\{\theta_1, \ldots, \theta_n\}$ of $\mathbb{R}^n$ which generates $\Lambda\mathbb{R}^n$ are refered to as the odd coordinate functions on (M, A).

By analogy with the previous example the universal measuring coalgebra $P(A(M), \mathbb{R})$ ought to provide a picture of the (dual to) the jet bundle for the graded manifold (M, A). It does. The only novelty is that in addition to conventional tangent vectors $\frac{\partial}{\partial x_i}$, corresponding to conventional coordinate functions on M, T^1_m contains odd tangent vectors $\frac{\partial}{\partial \theta_j}$ corresponding to the odd coordinate functions. The higher order tangent spaces contain suitable mixed derivatives of odd and ordinary coordinates. The result and the picture are still the same. The only novelty is that each T^k_m comes with a $\mathbb{Z}_2$ grading, $T^k_m = T^k_{m,0} + T^k_{m,1}$, and contains derivatives of both odd and even coordinates. (See Figure 2).

Result

$$P(A,\mathbb{R}) = \sum_{m\in M} \bigcup_k T^k_m.$$

This idea was exploited extensively by Kostant in his work on graded manifolds.

Figure 2:

5. Applications to the manifold of maps between manifolds

Let M and X be manifolds. It would seem that the above recipe for obtaining jet bundle information about the manifold $\mathcal{M}(X,M)$ of smooth maps from X to M would be a very unpromising one: to start with $\mathcal{M}(X,M)$, form $C^\infty(\mathcal{M}(X,M))$, and then $P(C^\infty(\mathcal{M}(X,M)),\mathbb{R})$ is not an attractive task. Nonetheless, measuring coalgebras do provide simple and easy access to the jet bundle of $\mathcal{M}(X,M)$ through the following trick.

There are two key observations.

1. If * is a point, then $\mathbb{R} = C^\infty(*)$.
2. $M = \mathcal{M}(*,M)$.

Thus, since the recipe

$$P(C^\infty(M),\mathbb{R}) = P(C^\infty(M),C^\infty(*)) \longrightarrow \mathrm{Hom}(C^\infty(M),C^\infty(*))$$

gives a satisfactory account of jet bundle information for $M = \mathcal{M}(*,M)$, it is reasonable to postulate

$$P(C(M),C^\infty(X)) \longrightarrow \mathrm{Hom}(C^\infty(M),C^\infty(X))$$

as a candidate for the dual jet bundle of $\mathcal{M}(X,M)$. In fact, if one defines the cocommutative part $P_c(C(M),C^\infty(X))$ of $P(C(M),C^\infty(X))$,

$$P_c(C^\infty(M),C^\infty(X)) = \{p \in P(C^\infty(M),C^\infty(X)) : \sum_{(p)} p_{(1)}\otimes p_{(2)} = \sum_{(p)} p_{(2)}\otimes p_{(1)}\}$$

then $P_c(C^\infty(M), C^\infty(X))$ does very well as a candidate for the dual jet bundle in that the picture is exactly the same as in the case of ordinary manifolds, and it contains most constructions reasonably expected to be elements of the dual jet bundle.

Result

i If $\sigma : X \longrightarrow M$ is a smooth map then $\mathbb{R}\sigma$ is a subcoalgebra of $P_c(C^\infty(M), C^\infty(X))$.

ii If $p : TM \longrightarrow M$ is the tangent manifold of M with its projection onto M, let $\tau : X \to TM$ be a smooth map with $p\tau = \sigma$. Then if $\mathbb{R}_\sigma + \mathbb{R}_\tau$ is the coalgebra of example **ii**, with σ point-like and τ primitive, $\mathbb{R}_\sigma + \mathbb{R}_\tau$ is a measuring subcoalgebra of $P_c(C^\infty(M), C^\infty(X))$. (See Figure 3).

iii

$$P_c(C^\infty(M), C^\infty(X)) = \sum_{\sigma : X \to M \text{ smooth}} \bigcup T^k_\sigma .$$

Here $T^0_\sigma = \mathbb{R}_\sigma$, and T^1_σ is the set of all derivations from $C^\infty(M)$ to $C^\infty(X)$ with respect to σ: that is all linear functionals τ satisfying

$$\tau(fg) = \tau(f)\sigma(g) + \sigma(f)\tau(g).$$

By inspecting the effect of composing τ with the evaluation at a point x in X, it can be seen that all such τ are maps from X to TM such that $\sigma = \pi\tau$. (See Figure 4).

Figure 3:

Property **iii** is a consequence of the fact that $P_c(C^\infty(M), C^\infty(X))$ is a cocommutative coalgebra all of whose simple subcoalgebras are pointlike. For a long time the attractive possibilities of including the not necessarily cocommutative parts of $P(C^\infty(M), C^\infty(X))$ were neglected. However, using the non cocommutative bits gives hope of representing quantum groups as "genuine" transformation groups. This brings the story up to my current interests. I will finish by giving one example of the type of non cocommutative measuring coalgebra which may be useful in representing quantum groups and related objects.

Figure 4:

6. A non cocommutative measuring coalgebra

Let $A = B = \mathbb{R}[z]$, the polynomial algebra over $\mathbb{C}$ on one generator. Let C be the coalgebra $C = \mathbb{R}E + \mathbb{R}_k + \mathbb{R}_h$ of example **iii** in the first section.

Define a linear map $\omega : C \to \mathrm{Hom}(\mathbb{R}[z], \mathbb{R}[z])$ by setting

$$\omega h(z) = 0, \quad \omega k(z) = z, \quad \omega E(z) = 1.$$

In order for ω to be a measuring map it must be that ωh and ωk are algebra homomorphisms, so that

$$\omega h(z^n) = 0, n > 0, \quad \omega h(1) = 1, \quad \omega k(z^n) = z^n \text{ for all } n.$$

Moreover inductively it can be shown that

$$\omega E(z^n) = z^n, n > 0, \quad \omega E(1) = 0.$$

It is curious that the absence of the coefficient one normally expects of differentiation results in the assymetry in the product rule (the measuring condition). However, the behaviour is typical of difference operators, as opposed to differential operators, and there are many variants.

It is also curious that the algebraic properties which enable such difference operators to exist in this case are

i) The image of $\omega k - \omega h$ is the ideal of $\mathbb{R}[z]$ generated by the single generator z, and

ii) any element of $\mathbb{R}[z]z$ can be written uniquely in the form $p(z)z$.

The interrelations between conformal field theories, loop algebras and quantum groups are an intriquing field of study. It is my belief that the algebraic properties above are at the root of the connection between quantum groups and conformal field theories.

References

[1] Batchelor, M. Measuring coalgebras, quantum group-like objects, and non-commutative geometry. In *Differential Geometric Methods in Theoretical Physics*, C. Bartocci, U. Bruzzo, R. Cianci (eds). Springer Lecture Notes in Physics 375, 1990.

[2] Batchelor, M. In search of the graded manifold of maps between graded manifolds. In *Complex Differential Geometry and Supermanifolds in Strings and Fields*, P.J.M. Bongaarts and R. Martini (eds), Springer Lecture Notes in Physics 311, 1988.

[3] Exton, *H. q-Hypergeometric Functions and Applications.* Ellis Horwood, 1983.

[4] Kostant, B. Graded Manifolds, Graded Lie Groups and Prequantisation. Springer Lecture Notes in Maths. 570. 1975.

[5] Lusztig, G. *Quantum deformations of certain simple modules over enveloping algebras.* Advances in Math.

[6] Sweedler, M. *Hopf Algebras.* Benjamin 1969.

Moduli Spaces

An Interdisciplinary Workshop organized by Sylvie Paycha

This session was of an experimental nature; three short twenty minutes talks were to be given on the topic of moduli spaces in relation to the different topics of the sessions of the conference, namely one by Caroline Series (given by Tan Lei since Caroline Series was unable to attend the conference) on moduli spaces and dynamical systems, one by Rosa Maria Miró–Roig on moduli spaces and classification problems in algebraic geometry, one by Sylvie Paycha on moduli spaces and quantum field theory. Laura Fainsilber spontaneously gave a talk on moduli spaces and number theory of which she kindly gave a brief account for these proceedings.

This interdisciplinary session on moduli space came out to be a success; it lead to lively discussions and even to a spontaneous talk (mentioned above). It was followed by a discussion session on moduli spaces where we tried to understand better the different points of views on moduli spaces presented in the different talks. There was clearly a need for such interdisciplinary discussions within this EWM conference in Madrid and we hope that further interdisciplinary sessions will be organised in the future!

Sylvie Paycha

Proc. of the 7^{th} EWM meeting, Madrid, 1995

Moduli spaces and conformal dynamics

Caroline Series

Warwick University, England
cms@maths.warwick.ac.uk

1. Introduction

To understand moduli we first have to understand the meaning of a *complex structure* on a surface. A *Riemann Surface* is a surface (i.e. a 2-manifold) S with local charts to $\mathbb{C}$ such that the overlap maps between charts are complex analytic. Remember that a complex analytic map locally preserves angles and expands or contracts distances. In fact, using Taylor series you see that locally it looks like

$$z \mapsto f(a) + f'(a)(z - a).$$

In other words, it is just an *affine* map $z \mapsto cz + d$ for some c, d. This map is a combination of a translation, rotation and similarity. A map like this which preserves angles is called *conformal*. (See Sylvie Paycha's paper). In fact a map of a subset of $\mathbb{C}$ is conformal if and only if it is complex analytic. Two surfaces S_1 and S_2 are said to be *conformally equivalent* if there is a homeomorphism $f : S_1 \mapsto S_2$ such that the maps f induces between charts are complex analytic (equivalently conformal). We can thus talk about the *conformal* or *complex structure* of a surface.

2. The problem of moduli

The problem of moduli is to describe the different conformal (or complex) structures there can be on a fixed topological surface.

Facts (which we are not going to prove.)

The complex structure of a surface of genus g can be described by $3g-3$ complex parameters (equivalently by $6g-6$ real parameters). The set of all possible moduli can be given, roughly speaking, the structure of a $3g-3$ dimensional complex manifold.

Note

There is often confusion between *Teichmüller space* and *Moduli space*. Both classify the possible complex structures on a surface. You get *Teichmüller space* when you fix what is called a *marking* on the surface. This means that on each of the two surfaces to be compared you specify that certain curves are generators of the fundamental group. When you are matching the two surfaces your homeomorphism is required to carry one set of marked generators into the equivalent set on the other surface. You get *Moduli space* when you ignore the marking and only ask for a homeomorphism between the surfaces, not worrying about what it does to the generating curves. Moduli space is a quotient of Teichmüller space by the action of the *Mapping Class Group*. This is the group of diffeomorphisms of the surface, modulo diffeomorphisms isotopic to the identity.

3. Two first examples

The first step in classifying conformal structures is the well known but still remarkable *Riemann Mapping Theorem*, first proved fully by Osgood in 1900. It states:

Any simply connected open subset $U \subset \mathbb{C}$ which has at least 2 points in its boundary is conformally equivalent to the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

In other words there is an analytic bijection between U and $\mathbb{D}$. (The inverse of an analytic map is automatically analytic.) Recently Sullivan and Rodin gave a very interesting and more constructive proof of this theorem which allow one, given the region U, to implement the map easily by computer.

The next simplest regions to classify are *annuli*. By definition, a (conformal) annulus is a bounded open subset of $\mathbb{C}$ whose complement has 2 connected components. By methods similar to those used in the Riemann mapping theorem it is proved (by Koebe)that any conformal annulus can be mapped by an analytic bijection to the region

$$\{z \in \mathbb{C} : 1/r < |z| < 1\} \text{ for some } r > 1 \quad (r \text{ might be infinity}).$$

Thus to solve the problem of moduli for annuli, we only have to determine when there is a conformal map between two of these standard regions.

Theorem 1 *If $r \neq s$, there is no complex analytic bijection between the regions $\{z \in \mathbb{C} : 1/r < |z| < 1\}$ and $\{z \in \mathbb{C} : 1/s < |z| < 1\}$.*

Proof

Suppose f were such a map. We may assume $|f(z)| \to 1$ as $|z| \to 1$. Let

$$g(z) = \log r \log f(z) - \log s \log z$$

Then $h(z) = \mathrm{Re} g(z)$ is *harmonic* and vanishes on the circles $|z| = 1$, $|z| = 1/r$. By the maximum principle, $h \equiv 0$. Therefore g is also constant (by the Cauchy Riemann equations). However, if we make a circuit round $|z| = 1/r$, g changes by

$$i[\log r \cdot 2\pi - \log s \cdot 2\pi].$$

Since g is constant the change is 0 and hence $r = s$.

The real number $\log r > 0$ associated to an annulus by this result is therefore unique and is called the *modulus* of the annulus. It completely specifies the annulus up to conformal homeomorphisms. The *moduli space* $\{\log r \in \mathbb{R} : r > 1\} =]0, \infty]$ does not quite fit into our picture of moduli space being a complex variety because an annulus is not a closed surface, neverthless it gives us a feel for what moduli are like.

4. Quasi-conformal maps

A very important idea in complex dynamics is the idea of a *quasi-conformal map*. This is a map which is not conformal but which stretches and shrinks with "bounded distortion". It is easy to write down a simple quasi-conformal map between annuli of different moduli – just map the annuli to their standard position and then stretch the smaller one evenly along radii to fit onto the larger. It is a fundamental theorem of Teichmüller that for *any* map between two surfaces, there is always a homotopic map of "least stretch" and that the surface can be divided into rectangles or annuli on each of which the map acts like one of these simple stretches.

5. The general case

The key to solving the moduli problem for a general topological surface is the famous *Uniformisation Theorem* proved by Koebe in 1907. It states that any simply connected Riemann surface is conformally equivalent to exactly one of $\mathbb{C}$, $\mathbb{C} \cup \infty$, $\mathbb{D}$.

We can use it to classify all Riemann surfaces by the following two facts, which are not hard to prove from the definitions:

1. The universal covering space of a Riemann surface has a complex structure and is again a Riemann surface.

2. The covering maps are complex analytic homeomorphisms.

Thus we have only to list all the possible covering maps in the three cases to find all possibilities. A covering map must be a fixed point free analytic homeomorphism of the universal cover onto itself. In the case of $\mathbb{C} \cup \infty$ there are no suitable analytic homeomorphisms because all of them have fixed points. In case of $\mathbb{D}$ the maps are exactly the linear fractional transformations which map the unit disc to itself without fixed point in the disc. These are the same as the isometries of 2-dimensional hyperbolic geometry. Thus it is possible to put a hyperbolic metric on the quotient surface and measure the moduli in terms of hyperbolic geometry measurements. This works whenever the surface we want has genus ≥ 2. If we had a bit more time we could explain exactly what the $6g - 6$ real measurements for a surface of genus g are. Finally in case of $\mathbb{C}$ the only fixed point free conformal homeomorphisms of $\mathbb{C}$ to itself are translations $z \mapsto z + b$, where b is complex. The most interesting case is when the covering group is $\mathbb{Z}^2$ and the quotient $\mathbb{C}/\mathbb{Z}^2$ is a torus. After scaling, translating and rotating (which are all analytic homeomorphisms) we can assume that the group G_τ of covering transformations is generated by $z \mapsto z + 1$ and $z \mapsto z + \tau$ for some complex τ with $\mathrm{Im}\tau > 0$. We have to find out when there is an analytic homeomorphism between two tori $\mathbb{C}/G_\tau$ and $\mathbb{C}/G_\sigma$. Suppose f were such a map. Let $\tilde{f}$ be the lift of f to $\mathbb{C}$. Then $\tilde{f}$ must be periodic relative to the lattice points $m + n\tau, m, n \in \mathbb{Z}$. It is not hard to show that the only possibilty is that f maps the lattice points $m + n\tau, m, n \in \mathbb{Z}$ to the points $q + r\sigma, q, r \in \mathbb{Z}$ and hence that $\tau = \frac{a\sigma+b}{c\sigma+d}$ for some integers a, b, c, d with $ad - bc = 1$. This gives us the well known picture that the moduli space for tori is the upper half plane (the τ plane) quotiented by the action of the group $SL(2, \mathbb{Z})$. (See Rosa Maria Miró Roig's paper).

A good reference for this material is *G. Jones and D. Singerman, Complex functions, Cambridge University Press, 1987.*

6. Moduli in complex dynamics

How do moduli arise in complex dynamics? In complex dynamics one is studying a rational function f mapping the Riemann sphere $\mathbb{C} \cup \infty$ to itself . The Riemann sphere divides into the Julia set, on which the grand orbits of f are dense, and the Fatou set, on which the grand orbits are either discrete (in attracting or parabolic basins) or leaves of dynamically defined foliations (in Siegel discs, Herman rings and superattracting basins). Pick a connected component Ω_0 of the Fatou set on which the action of the map is discrete. It makes sense to form the quotient Ω_0/f . This is the orbit space in which all points in the same f orbit are identified. Because the map was acting discretely, the quotient is a Riemann sufrace, usually with some branch points. A lot of use has been made of the idea that we can relate different rational maps by deforming these quotient surfaces by using quasi-conformal maps. We can also investigate the space of all maps f with a given combinatorial structure by looking at the moduli spaces of the quotients. Sullivan proved a very fundamental theorem, the *non-wandering domain theorem*, by using the fact that the moduli spaces for each quotient are finite dimensional. In my own work the rational map f is replaced by a group Γ of linear fractional transformations. Looking at where the group orbits are discrete or not, the Riemann sphere still splits into a "Julia set" and a "Fatou set". The quotients Ω_0/Γ are Riemann surfaces. We are interested in studying the relation between the moduli of the surface and the complex parameters which go into defining the generators of the group. This gives some very interesting and concrete pictures of Teichmüller space.

7. An application to complex dynamics

We end with a nice application of the moduli of annuli to complex dynamics. In studying the Mandelbrot set for cubic polynomials, Branner and Hubbard had a situation in which they had an infinite set of nested annuli, and they needed a criterion for whether the intersection consisted of one or many points. They used 2 basic results about annuli:

1. Grötzsch's inequality: Suppose a sequence A_n of open annuli are nested inside an open annulus A, each winding once around the hole in A. Then

   $\sum mod A_n \leq mod A$

2. If A is an open bounded annulus of infinite modulus the bounded component of the complement (i.e. the hole) consists of one point.

   Thus if you can find a sequence A_n of nested annuli such that the sum of their moduli is infinite, you know that the hole in the middle of the nest consists of just one point.

Proc. of the 7^{th} EWM meeting, Madrid, 1995

Moduli spaces and path integrals in Quantum Field Theory

Sylvie Paycha

Université Blaise Pascal, Aubiere, France
paycha@ucfma.univ-bpclermont.fr

Warning: *These notes are informal notes which are only meant to give a hint as to the variety ot topics the investigation of moduli spaces in relation to quantum field theory can lead to.*

1. Moduli space of a Riemann surface

This section is closely related to Caroline Series' paper on moduli spaces and the reader is referred to the references therin concerning the contents of this section.

Let Λ be a smooth compact surface without boundary of genus p. The purpose of this first section is to define the moduli space of such a surface. We shall need the notions of Riemannian metric and conformal structure.

Riemannian metrics Let p be a point on Λ and let (x^1, x^2) be a system of local coordinates at point p on Λ. A *Riemannian metric* on Λ at point p is defined locally at point p by a symmetric two by two matrix (g_{ab}) with strictly positive determinant. A Riemannian metric is a global object (a covariant symmetric two tensor on Λ) but its local description depends on the choice of local coordinates. If f is a diffeomorphism of the surface Λ that takes (x^1, x^2) around p to another local system of coordinates (y^1, y^2) around $f(p)$, then locally $g_{a,b}$ transforms to

$$(f^*g)_{ab} \equiv g'_{ab}$$

where

$$g'_{ab} = \sum_{c,d=1,2} g_{cd} \frac{dx^c}{dy^a} \frac{dx^d}{dy^b}.$$

This defines an action of the group $Diff(\Lambda)$ of diffeomorphisms on the space $Met(\Lambda)$ of Riemannian metrics on Λ:

$$\begin{array}{rccc} \alpha: & Diff(\Lambda) \times Met(\Lambda) & \longrightarrow & Met(\Lambda) \\ & (f, g) & \longmapsto & f^*g. \end{array}$$

The metric is a tool to measure lengths of curves and areas of surfaces on the manifold. In particular, since Λ is compact, we can define the area of Λ for a given metric g by

$$A_g(\Lambda) \equiv \int_\Lambda \sqrt{detg}$$

as well as the area

$$A(x,g) \equiv A_{x^*g}(\Lambda) \tag{$*$}$$

of $x(\Lambda)$ where x is a map that embeds Λ in $\mathbb{R}^d$.

One can stretch or shrink the metric by multiplying it by a constant factor $g \to kg$, $k > 0$ thus multiplying the area $A_g(\Lambda)$ by the same factor k. One can also multiply the metric pointwise $g(p) \to (e^\phi g)(p) \equiv e^{\phi(p)} g(p)$ by a strictly positive function on Λ. Letting $Met(\Lambda)$ denote the space of Riemannian metrics on Λ (it is an infinite dimensional space) and setting $W(\Lambda) \equiv \{e^\phi, \phi \in Map(\Lambda, \mathbb{R})\}$, we thus define an action of $W(\Lambda)$ on $Met(\Lambda)$:

$$\begin{array}{rccc} \beta: & W(\Lambda) \times Met(\Lambda) & \longrightarrow & Met(\Lambda) \\ & (e^\phi, g) & \longmapsto & e^\phi \cdot g. \end{array}$$

The *conformal class* of a given metric g is the set of Riemannian metrics

$$[g] \equiv \{e^\phi g, \phi \in Map(\Lambda, \mathbb{R})\}$$

.

A *conformal transformation* of Λ is a diffeomorphism f of Λ which preserves the conformal class of the metric, i.e such that $f^* g = e^\phi g$ where ϕ is a real function on Λ.

Given a metric g on Λ and a local coordinate system (x^1, x^2) at point $p \in \Lambda$, one can diagonalise the metric and thus find another system of coordinates (y^1, y^2) in which the metric matrix takes the form $\begin{bmatrix} e^{\phi(p)}\lambda_1 & 0 \\ 0 & e^{\phi(p)}\lambda_2 \end{bmatrix}$ where $\lambda_1, \lambda_2 \in \mathbb{R}$. Then any metric $e^\phi g$ in the conformal class of g is also diagonal in this coordinate system and its matrix takes the form $\begin{bmatrix} e^{\phi}\lambda_1 & 0 \\ 0 & e^{\phi}\lambda_2 \end{bmatrix}$. The system of coordinates (y^1, y^2) is called *isothermal* for the conformal class $[g]$. Setting $z \equiv y^1 + iy^2$, one can equip Λ with a *complex structure*. There is in fact a one to one correspondence between the set $C(\Lambda)$ of complex and the set $Conf(\Lambda)$ of conformal structures on Λ .

We finally have the following isomorphisms:

$$Met(\Lambda)/W(\Lambda) \simeq C(\Lambda) \simeq Conf(\Lambda).$$

The action α of the diffeomorphism group on $Met(\Lambda)$ induces an action on $Conf(\Lambda)$ (and hence on $C(\Lambda)$), setting for $f \in Diff(\Lambda)$, $f^*[g] = [f^* g]$. In general there are diffeomorphisms which do not preserve a given complex (or conformal) structure; they take a given complex (or conformal) structure to another one (see Caroline's talk). These two complex (or conformal) structures are then called equivalent.

The *moduli space* $Mod(\Lambda)$ of Λ is the set of non equivalent complex (or conformal) structures on Λ. It can be described as a quotient space in four equivalent ways:

$$\begin{aligned} C(\Lambda)/Diff(\Lambda) &\simeq Conf(\Lambda)/Diff(\Lambda) \\ &\simeq Met(\Lambda)/W(\Lambda)/Diff(\Lambda) \\ &\simeq Met(\Lambda)/Diff(\Lambda)/W(\Lambda) \end{aligned}$$

where $Diff(\Lambda)$ acts via the action α on the space of metrics and $W(\Lambda)$ via the action β.

In what follows, we want to interpret the moduli space $Mod(\Lambda)$ in the framework of string theory in terms of the quotient space of a space of paths by the action of a symmetry group.

2. Moduli space and path integrals

The space of paths

Here, we shall see Λ together with a fixed Riemannian metric g on Λ and an embedding $x : \Lambda \to \mathbb{R}^d$ as a path descibed by a *loop (or closed string)* moving in space time $\mathbb{R}^d$. The starting point and the end point of its trajectory, which would introduce boundaries to the surface described by the loop are taken at infinitely distant times so that we can assume everything happens as though the surface Λ were boundaryless and compact! If we consider several interacting loops, i.e loops meeting up together and splitting again, the resulting surface can be of any genus $p > 0$.

The space $\mathcal{P}$ of paths can therefore be seen as the product space $Emb(\Lambda) \times Met(\Lambda)$ of the space $Emb(\Lambda)$ of embeddings of Λ into $\mathbb{R}^d$ with the space of Riemannian metrics on a boundaryless compact smooth surface Λ of any genus.

D$iff(\Lambda)$ as a symmetry group

A classical string evolves acording to a *minimal energy principle* and describes a surface with minimal area $A(x,g)$ as defined in $(*)$. This area is also called the *classical action or energy.*

Since the area does not depend on the chosen parametrization of the surface Λ, this action $A(x,g)$ is invariant under the action of the diffeomorphism group $Diff(\Lambda)$, i.e

$$A(x \circ f, f^* g) = A(x,g).$$

We see this group as a *symmetry group* for the classical action or energy.

In fact, this action is also invariant under the action of $W(\Lambda)$ so that the symmetry group Sym is in fact a larger group, the smallest one containing both $W(\Lambda)$ and $Diff(\Lambda)$, which we shall not describe in detail here.

Moduli space

Diffeomorphisms act trivially on $Emb(\Lambda)$ via composition ($x \to x \circ f, f \in Diff(\Lambda), x \in Emb(\Lambda)$) and $W(\Lambda)$ does not act on $Emb(\Lambda)$ so that the action of $Sym(\Lambda)$ on $Emb(\Lambda)$ reduces to that of $Diff(\Lambda)$. Combining this action with the action of $Sym(\Lambda)$ on the space of Riemannian metrics, one can define an action of the symmetry group Sym on the space of paths $\mathcal{P} = Emb(\Lambda) \times Met(\Lambda)$.

We shall call two paths equivalent when there is an element of the symmetry group that transforms one into the other. The space of non equivalent paths is thus described by

$$\text{“}\mathcal{P}/Sym = (Emb(\Lambda)/Diff(\Lambda)) \times (Met(\Lambda)/Sym(\Lambda)) = (Emb(\Lambda)/Diff(\Lambda)) \times Mod(\Lambda)\text{”}$$

Since the moduli space coincides with $Met(\Lambda)/Sym(\Lambda)$, we see how it arises here as a subspace of the *space of non equivalent paths* $\mathcal{P}/Sym(\Lambda)$.

Path integrals

In quantum field theory one cannot "see" paths but only "mean values" over the space of paths, namely observables. An *observable* is the mean value $< O >$ of a function $O : \Lambda \to \mathbb{R}$ with respect to a formal measure on the space of paths:

$$< O > \equiv Z^{-1} \int_{\mathcal{P}} O(p) e^{-A(p)} dp$$

where $A(p)$ is the classical action (or energy) of the theory-in the case of strings it is the area $A(x,g)$ given in $(*)$- "dp" is a formal volume measure on the infinite dimensional space of paths $\mathcal{P}$, and Z is the normalising constant $Z = \int_{\mathcal{P}} e^{-A(p)} dp$.

Giving a meaning to such integrals on infinite dimensional manifolds is extremely difficult and the description of path integrals gives rise to problems of a geometric or topological nature, which a priori can look very different from the original one, such as looking for invariants of manifolds [D] (see also the recent work by Seiberg and Witten on the subject reviewed in [B])

In the case of strings mentioned above, an interpretation of the partition function

$$\text{“}Z = \int_{Emb(\Lambda)\times Met(\Lambda)} e^{-S(x,g)} \quad dxdg''$$

was first suggested in [P] (see also [Po] about this interpretation) and further investigated by many authors using algebraic-geometric techniques (see e.g [Bo], [Ph] ,[S])

Anomalies

Whenever the function O is invariant under the symmetry group, one can hope to *reduce the path integral* given by $< O >$ to an integral on the moduli space (which is sometimes finite dimensional as in the case of strings (see C.Series' talk)), when seen as a quotient of the space of paths via the action of the symmetry group, since the latter leaves the classical energy $A(p)$ arising in this integral invariant.

However, there can be an obstruction to doing this if the "formal volume measure" denoted by "dp" on the path space is not invariant under the action of the symmetry group. This is the case in the example considered above where "dg" is not invariant under $Sym(\Lambda)$; the symmetry we have at the classical level ($S(x,g)$ is invariant under $Sym(\Lambda)$ is " broken" at the quantised level when "integrating" w.r.to the "measure " "dx dg" on the space of paths.

This gives rise to *anomalies*, which can be interpreted as topological and geometric obstructions on a certain line bundle built on moduli space, namely a determinant bundle (see e.g [ASZ],[Bo],[BF],[F]). The fact that there is a *determinant line bundle* involved is related to the fact that a transformation of the above "path integral" via the action of the group gives rise to a jacobian determinant, also called the *Faddeev-Popov* determinant [BV]. In the case of stings mentioned above, the (conformal) anomaly arises from the non invariance of the formal measure "dg" on the space of metrics under the action of $W(\Lambda)$ [F].

References

[ASZ] O.Alvarez,I.M.Singer, B.Zumino, *Gravitational anomalies and the family's index theorem*, Comm. Math. Phys. **96** (1984), 409-417

[B] D. Bennequin, *Monopoles de Seiberg-Witten et conjecture de Thom*, Séminaire Bourbaki, **807** (Nov.1995)

[Bo] J.B.Bost, *Fibrés déterminants, déterminants= régularisés et mesures sur les espaces de modules des courbes complexes*, Séminaire Bourbaki, exposé 676, Astérisque 152-153 (1987), 113-149

[BF] J.M. Bismut, D.Freed, *The Analysis of elliptic families I* , Comm. Math. Phys. **106** (1986) ,157-176

[BV] O. Babelon, C.M. Viallet, *The geometrical interpretation of the Faddeev-Popov determinant*, Phys. Lett. **B85** (1979), 246-248

[D] S.K. Donaldson, *An application of gauge theory to the topology of 4-manifolds*, Journ. Diff. geom. 18 (1983), 279-315

[F] D.Freed,*Determinants, torsion and strings*, Comm. Math. Phys. **107** (1986), 483-513

[P] A.M. Polyakov, *Quantum geometry of bosonic strings*, Phys.Lett.B 103 (1981)

[Ph] D.H.Phong, *Complex Geometry and Strings*, Proceedings of Syposia in Pure Mathematics, Vol **54** Part 2 (1993), 467-527

[Pol] J.Polchinski, *From Polyakov to Moduli*, in "Mathematical aspects of string theory", S.T. Yau (ed.), World Scientific, Singapore, 1987

[S] D.J.Smit, *String theory and algebraic geometry of moduli spaces*, Comm.Math.Phys. **114**, p.645-685 (1988)

Proc. of the 7^{th} EWM meeting, Madrid, 1995

Notes on moduli spaces in Algebraic Geometry

Rosa M. Miró–Roig

Universitat de Barcelona, Spain
miro@cerber.ub.es

1. Introduction

Moduli spaces are one of the fundamental constructions of Algebraic Geometry. They arise in connection with classification problems and, although it is a fairly delicate subject, I will try to describe it in an elementary fashion.

Roughly speaking a moduli space for a collection of objects A and an equivalence relation $\sim$ is a classification space, i.e. a space (in some sense of the word) such that each point corresponds to one and only one equivalence class of objects. Therefore, as a set, we define the moduli space as equivalence classes of objects $A/\sim$. In our setting the objects are algebraic objects, and because of this we want an algebraic structure on our classification set. Finally we want our moduli space to be unique (up to isomorphism).

So the basic ingredients for a moduli problem are a collection of objects A, an equivalence relation $\sim$ on A and a concept of family of objects of A parametrized by an algebraic variety (or a scheme) S. (See the lectures on Algebraic Geometry for the precise definiton of algebraic variety and scheme.) Such a family consists of a collection of objects X_s, one for each $s \in S$, which fit together in some way corresponding to the structure of S. The precise definition of family depends on the particular moduli problem; however, in all cases, it satisfies the following properties:

1. A family parametrized by the variety $\{pt\}$ (consisting of a single point) is a single object of A.

2. There is a notion of equivalence of families parametrized by any given variety S, which gives the equivalence relation on A when $S = \{pt\}$; we denote this relation again by $\sim$.

3. For any morphism $\varphi : S' \to S$ and any family X parametrized by S, there is an induced family $\varphi^* X$ parametrized by S'. Furthermore we have the functorial properties: 1_S^* =identity, $(\varphi\psi)^* = \psi^*\varphi^*$; and it is compatible with $\sim$, i.e. if $X \sim X'$ then $\varphi^* X \sim \varphi^* X'$.

[0]These notes are intended to support our cross disciplinary discussion on moduli spaces. Many people have made important contributions without even being mentioned here. If I have made wrong attributions I apologize for that. In no case do I claim that any idea here at all originates from me.

The moduli problem consists in giving to $A/\sim$ a structure of algebraic variety which reflects the structure of families of objects of A.

2. Fine moduli spaces, coarse moduli spaces and quotients

We consider the contravariant functor $\mathcal{F} : (Varieties) \to (Sets)$, $S \longmapsto \mathcal{F}(S)$, where $\mathcal{F}(S)$ is the set of equivalence classes of families parametrized by the variety S. Suppose that M is an algebraic variety with underlying set is $A/\sim$. For any family X parametrized by S, we denote by $\mu_X : S \to M$ the map given by $\mu_X(s) := [X_s]$ where $[X_s]$ is the equivalence class of the object X_s.

Definition A fine moduli space for a given classification problem is a pair (M, Ψ) which represents the functor $\mathcal{F}$.

Let M be a variety and $\Psi : \mathcal{F} \to Hom(., M)$ the natural transformation given by $\Psi(S)(X) = \mu_X$ for any variety S and any family X parametrized by S. By definition the pair (M, Ψ) represents the functor $\mathcal{F}$ if Ψ is an isomorphism of functors.

Remark 1. From the definition it easily follows that the underlying set of the variety M is $A/\sim$.

Remark 2. If a fine moduli space exists for a given classification problem, then it is unique (up to isomorphism).

The morphism $1_M \in Hom(M, M)$ determines, up to the equivalence relation $\sim$, a family $U \in \mathcal{F}(M)$ which gives us the following alternative definition:

Definition A fine moduli space for a given classification problem consists of a variety M and a family U parametrized by M such that, for every family X parametrized by S, there is a unique morphism $\Psi : S \to M$ with $X \sim \Psi^* U$. The family U is called a universal family for the given problem.

Unfortunately there are very few classification problems for which a fine moduli space exists and it is necessary to find some weaker conditions which nevertheless determines a unique structure of algebraic variety on M.

Definition A coarse moduli space for a given classification problem consists of a variety M together with a natural transformation $\Psi : \mathcal{F} \to Hom(., M)$ verifying:

1. $\Psi(pt)$ is bijective,
2. For any variety N and any natural transformation $\varphi : \mathcal{F} \to Hom(., N)$, there exists a unique natural transformation $\tau : Hom(., M) \to Hom(., N)$ such that $\varphi = \tau\Psi$.

Remark 3. From the definiton it easily follows that M is a variety with underlying set $A/\sim$.

Remark 4. If a coarse moduli space exists for a given classification problem, then it is unique (up to isomorphism).

Remark 5. A fine moduli space for a given classification problem is always a coarse moduli space for this problem but, in general, not vice versa. In fact, there is no a priori reason why the map $\Psi(S) : \mathcal{F}(S) \to Hom(S, M)$ should be bijective for varieties S other than $\{pt\}$.

In the last part of this section we introduce the notion of categorial quotient of a variety by the action of a group and we will explain its connection with moduli problems.

Definition Let G be an algebraic group acting on a variety X. A categorial quotient of X by G is a pair (Y, φ) where Y is a variety and $\varphi : X \rightarrow Y$ is a morphism such that:

1. φ is constant on the orbits $o(x) = \{gx; g \in G\} \subset X$ of the action;

2. For any variety Z and for any morphism $\psi : X \rightarrow Z$ which is constant on orbits, there is a unique morphism $g : Y \rightarrow Z$ such that $g\varphi = \psi$.

If in addition $\varphi^{-1}(y)$ consists of a single orbit for all $y \in Y$, we call (Y, φ) an orbit space.

Remark 6. A categorial quotient is unique up to isomorphism and it exists in general circumstances. (For instance, if G is a reductive group acting on an affine variety X; or if G is a reductive group acting on a projective variety X and we restrict our attention to the open subset $X^{ss} \subset X$ of semistable points of X.)

To relate categorial quotients to moduli spaces we need to introduce some extra definitions.

Definition For a given moduli problem, a family X parametrized by a variety S is said to have the local universal property if for any family X' parametrized by S' and any point $s \in S'$, there exists a neighbourhood U of s such that $X'_{|U}$ is equivalent to the family induced from X by some morphism $U \rightarrow S$.

Proposition 1 *Suppose that, for a given moduli problem, there exists a family X parametrized by S having the local universal property. Suppose that a group G acts on S and that $X_s \sim X_t$ if and only if s and t belongs to the same orbit of this action. Then:*

1. *Any coarse moduli space is a categorial quotient of S by G;*

2. *A categorial quotient of S by G is a coarse moduli space if and only if it is an orbit space.*

More details on fine moduli spaces, coarse moduli spaces and quotient can be found, for instance, in [MFK], [MS] or [N].

3. Examples

As particular examples of moduli problems, we will briefly discuss the three following cases:

1. Hilbert schemes;

2. The moduli space for the isomorphism classes of smooth curves of genus g;

3. The moduli space for stable vector bundles with given Chern classes on a projective variety X.

Example 1

Let $\mathbb{P}^r$ be the r-dimensional projective space over a field $\mathbf{k}$. The first classification problem that we will deal with is the classification problem for closed projective subschemes $X \subset \mathbb{P}^r$ and we will see that there exists a fine moduli space for such a classification problem. Roughly speaking our objects will be closed subschemes of $\mathbb{P}^r$ with given Hilbert polynomial $p(t) \in \mathbb{Q}[t]$, the equivalence relation will be the equality and we have the following notion of family:

Definition A flat family of closed subschemes of $\mathbb{P}^r$ parametrized by a $\mathbf{k}$-scheme S is a closed subscheme $\mathcal{X} \subset \mathbb{P}^r_S = \mathbb{P}^r$x S such that the morphism $\mathcal{X} \to S$ induced by the projection $\mathbb{P}^r_S = \mathbb{P}^r$x $S \to S$ is flat.

It is an important fact that flat families $\mathbb{P}^r_S = \mathbb{P}^r$x $S \supset \mathcal{X} \to S$ of closed subschemes of $\mathbb{P}^r$ parametrized by a connected $\mathbf{k}$-scheme S have all their fibres with the same Hilbert polynomial.

We fix an integer r and a polynomial of the form $p(t) = \sum_{i=0}^r a_i \binom{t+r}{i} \in \mathbb{Q}[t]$ where a_i's are integers. We consider the contravariant functor

$$\underline{Hilb}^r_{p(t)} : (k-schemes) \to (Sets)$$

where $\underline{Hilb}^r_{p(t)}(S) := \{$flat families of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$ parametrized by $S\}$. In 1960, A. Grothendieck proved (See [G] or [M]):

There is a unique projective scheme $Hilb^r_{p(t)}$ which parametrizes a flat family, $\mathbb{P}^r$ x $Hilb^r_{p(t)} \supset \mathcal{W} \xrightarrow{\pi} Hilb^r_{p(t)}$, of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, and having the following universal property: for every flat family, $\mathbb{P}^r$ x $S \supset \mathcal{X} \xrightarrow{f} S$, of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, there is a unique morphism $g : S \to Hilb^r_{p(t)}$, called the classification map for the family f, such that π induces f by base change; i.e. $\mathcal{X} = S$ x $_{Hilb^r_{p(t)}} \mathcal{W}$. Following the definiton of fine moduli space π is called the universal family.

In the usual language of categories the pair $(Hilb^r_{p(t)}, \pi)$ represents the functor $\underline{Hilb}^r_{p(t)}$ and the classification problem for projective subschemes has a fine moduli space.

Since any closed subscheme of $\mathbb{P}^r$ with Hilbert polynomial $p(t) = \binom{t+n}{n}$ is a linear subspace of $\mathbb{P}^r$ of dimension n, we have:

$$Hilb^r_{\binom{t+n}{n}} = Grass(n+1, r+1).$$

Hence the Hilbert schemes can be considered as generalizations of the grassmannians; i.e. varieties parametrizing all (n+1)-dimensional vector subspaces of a given (r+1)-dimensional vector space V.

Remark 7: We know the existence of the Hilbert scheme but its local and global properties are very far from being understood; even for the case of projective space curves. I will not discuss here either recent contributions or open problems.

Example 2

We consider the set $\{C_\alpha\}$ of smooth curves of genus g and we ask whether the set M_g of such curves, up to isomorphism, may be given the structure of an algebraic variety in a natural way (such is the case for g=1, where the j-invariants form an affine line). To begin with, we define a flat *family* of smooth curves of genus g with base S to be a variety $\mathcal{V}$ and a flat morphism $\pi : \mathcal{V} \to S$ such that for each point $s \in S$ the fiber $\mathcal{V}_s := \pi^{-1}(s)$ is isomorphic to C_α for some α. The best way to specify the algebraic structure on M_g would be to require it to be a universal parameter variety for families of curves of genus g, in the following sense: we require that there be a flat family $\mathcal{X} \to M_g$ of smooth curves of genus g such that for any other flat family $f : \mathcal{V} \to T$ of smooth curves of genus g, there is a unique morphism $g : T \to M_g$ such that $\mathcal{V} = g^*\mathcal{X}$. In this case we call M_g a fine moduli space for the curves $\{C_\alpha\}$.

Unfortunately, there are very few classification problems for which a fine moduli space exists. One of the reasons why the universal family $\mathcal{X} \to M_g$ does not exist is that there are nontrivial flat families of smooth curves of genus g, all whose fibres are isomorphic to each other. Hence it is necessary to find some weaker condition which nevertheless determines a unique algebraic structure on M_g. This problem was settled by D. Mumford. He proved that for $g \geq 2$ there is a coarse moduli space M_g which has the following properties (See [M]; Theorem 5.11):

1. The set of closed points of M_g is in one-to-one correspondence with the set of isomorphism classes of smooth curves of genus g;
2. if $f : \mathcal{V} \to T$ is a flat family of smooth curves of genus g, then there is a morphism $g : T \to M_g$ such that for each closed point $t \in T$, $\mathcal{V}_t$ is in the isomorphism class of curves determined by the point $g(t) \in M_g$.

Since all smooth connected curves of genus g=0 are isomorphic to $\mathbb{P}^1$, we have $M_0 = \{pt\}$. In case g=1, the j-invariants of elliptic curves define an affine line which is a coarse moduli space M_1 for the family of elliptic curves. In 1969, P. Deligne and D. Mumford proved that M_g for $g \geq 2$ is an irreducible, quasi-projective variety of dimension 3g-3 (See [DM]). An excellent discussion of this construction is given in [MFK], which includes references to other examples of the applications of geometric invariant theory as well.

Remark 8: The dimension of M_g was already stated by Riemann in his celebrated paper "Theorie der Abel'schen Functionen" of 1857 (See [R]). The word "moduli" is due to him and the subject has its origins in the theory of elliptic functions. The irreducibility was already observed by Klein (See [K]) and follows from results of Lüroth and Clebsch; and it was only much later that Baily showed that M_g has a natural structure of quasi-projective normal variety of dimension 3g-3.

Remark 9: Nowadays there are three principal approaches to constructing M_g.

1. In a transcendental setting, it can be constructed as a quotient of the Teichmüller space,
2. As a subvariety of a quotient of the Siegel upper half-space, or
3. Using Mumfords's geometric invariant theory, it can be constructed as a quotient of a Hilbert scheme.

The moduli space M_g is not compact, i.e., it is a quasi-projective variety but not a projective variety. The right choice of boundary points for M_g was discovered by Mayer and Mumford, in 1963, in an unpublished work. For an excellent discussion of M_g and its compactification, the reader could look at [MFK].

Example 3

As last example we will consider the moduli problem for vector bundles on a smooth projective variety X. We consider the set $A/\sim$ of isomorphism classes of rank r vector bundles on X with fixed Hilbert polynomial $H(m) \in \mathbb{Q}[m]$ and we would like to endow $A/\sim$ with a natural structure of scheme. To this end we define a family of vector bundles of rank r and Hilbert polynomial $H(m)$ parametrized by a $\mathbf{k}$-scheme S as a vector bundle $\mathcal{E}$ on S x X such that for all $s \in S$, $\mathcal{E}(s)$ is a rank r vector bundle on $X \cong \{s\}$ x X with Hilbert polynomial $H(m)$.

Unfortunately this moduli problem has no solution and to get at least a coarse moduli space we must somehow restrict the class of vector bundles that we consider. What kind of subfamily should be taken? In [MA], [MA1], M. Maruyama found an answer to this question: stable vector bundles. He proved: Let X be a smooth projective variety and let $A/\sim$ the set of isomorphism classes of stable vector bundle on X of rank r and Hilbert polynomial $H(m)$. Then, there is a coarse moduli scheme M which is a separated scheme, locally of finite type. This means:

1. The closed points of M are in one to one correspondence with the elements of $A/\sim$;
2. Whenever $\mathcal{F}$ is a flat family of vector bundles of $A/\sim$, parametrized by a scheme T (i.e. $\mathcal{F}$ is a vector bundle on X x T, flat over T, whose fibres are in $A/\sim$), then there is a morphism $\psi : T \rightarrow M$ such that for each closed point $t \in T$, $\psi(t)$ is the point of M corresponding to the class of the vector bundle $\mathcal{F}_t$ which is the fibre of $\mathcal{F}$ ovet t;
3. The morphism ψ can be assigned functorially; and
4. M is universal with the properties (1) and (2).

Remark 10: In spite of the great progress made during the last decades on the moduli spaces of vector bundles on smooth projective varieties (essentially in the framework of the Geometric Invariant Theory by Mumford), very litle is known about their local and global structure.

References

[B] W. Baily, *On the theory of Θ-functions, the moduli of abelian varieties and the moduli of curves*, Ann. of Math., **75** (1962), 342-381.

[DM] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, IHES Publ. Math. **36** (1969), 75-110.

[G] A. Grothendick, *Les schémas de Hilbert*, Sem. Bourbaki **221** (1960).

[H] J. Harris *On the Kodaira dimension of the moduli space of curves II*, Inv. Math. **75** (1984), 437-466.

[HM] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves I*, Inv. Math., **67** (1982), 23-86

[Ha] R. Hartshorne *Algebraic Geometry*, GTM **52** (1977)

[K] F. Klein, *Uber Riemann's Theorie der Algebraischen Functionen un Ihrer Integrale*, Teubner, Leipzig, (1882)

[MA] M. Maruyama, *Moduli of stable sheaves, I* , J. Math. Kyoto Univ., **17** (1977), 91-126

[MA1] M. Maruyama, *Moduli of stable sheaves, II*, J. Math. Kyoto Univ., **18** (1978), 557-614.

[M] D. Mumford, *Lectures on Curves on an Algebraic Surface*, Princeton Univ. Press, **59** (1966).

[M1] D. Mumford, *Geometric Invariant Theory*, Springer-Verlag, (1994)

[MFK] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, Springer-Verlag, (1965).

[MS] D. Mumford and K. Suominen, *Introduction to the Theory of moduli*, Prooc. fifth Nordic Summer School in Math., Oslo, (1970), 171-222.

[N] P.E. Newstead, *Introduction to moduli problems and orbit spaces*, Tata Inst. of Fundamental Research, Bombay, 1978.

[R] B. Riemann, *Theorie der Abel'schen Functionen*, Crelle J., **54** (1857), 115-155.

Proc. of the 7^{th} EWM
meeting, Madrid, 1995

An example of moduli spaces in number theory

Laura Fainsilber

Université de Franche – Comté, France
`laura@math.univ-fcomte.fr`

I thank Leila Schneps for introducing me to the subject of Grothendieck-Teichmüller theory, for explaining it to me, and for crucial help in preparing this text.

1. The inverse problem in Galois theory

The basic object of interest to number theorists is the field of rational numbers $\mathbb{Q}$, along with its field extensions. We will first consider finite Galois extensions of $\mathbb{Q}$: given an irreducible polynomial $P \in \mathbb{Q}[T]$, with roots α_i in an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, we let K_α be the smallest field containing $\mathbb{Q}$ and all the α_i. Such a field is called a Galois extension of $\mathbb{Q}$. The Galois group $G = Gal(K_\alpha/\mathbb{Q}) = Aut_{\mathbb{Q}}(K_\alpha)$ is defined to be the group of field automorphisms of K_α that fix the elements of $\mathbb{Q}$. It is a finite group, and the fundamental theorem of Galois theory asserts that there is a bijective correspondence between subfields of K_α which are Galois extensions of $\mathbb{Q}$, and normal subgroups of G. Namely, a normal subgroup H of G corresponds to the subfield L of elements of K_α fixed by all the automorphisms in H, and $Gal(L/\mathbb{Q}) = G/H$. In that case, K_α is also a Galois extension of L, with Galois group H. We express this in the following diagram:

$$\begin{array}{c} K_\alpha \\ | \\ L \\ | \\ \mathbb{Q} \end{array} \quad \left.\begin{array}{c} H \\ \\ G/H \end{array}\right\} G$$

The inverse problem in Galois theory consists of finding out which groups can be Galois groups of extensions of $\mathbb{Q}$, and constructing explicit extensions with such groups as Galois groups. For example, all finite abelian groups, the symmetric groups S_n and the alternating groups A_n can be Galois groups, as can the 26 sporadic simple groups (except maybe the Mathieu group M_{23}). With a combination of methods, many families of groups have been shown to be Galois groups. E. Noether found algebraic geometric conditions under which a group is a Galois group. More recently, Matzat and Mestre, among others, have worked on constructive approaches and given explicit polynomials that define Galois extensions for new families of groups. One conjectures, and most experts believe, that all finite groups are Galois groups over $\mathbb{Q}$.

2. Grothendieck's approach

In the last few years, a novel top-down approach has been developed, in particular by Drinfel'd, Ihara, Schneps, Lochak, inspired by ideas of Grothendieck. It consists of studying directly the group $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) = Aut_{\mathbb{Q}}(\overline{\mathbb{Q}})$. We extend our notion of Galois extensions to infinite algebraic extensions : we see a finite Galois extension K of $\mathbb{Q}$ as a subfield of $\overline{\mathbb{Q}}$, and see the Galois group of K as a finite quotient of $G_{\mathbb{Q}}$. The inverse problem in Galois theory is now to see which finite groups are quotients of $G_{\mathbb{Q}}$, and in general to describe the structure of the large and mysterious group $G_{\mathbb{Q}}$.

$$\begin{array}{ll} \overline{\mathbb{Q}} & \\ \mid \; G_{\infty} & \left.\right\} \\ K & \quad G_{\mathbb{Q}} \\ \mid \;\; G & \left.\right\} \\ \mathbb{Q} & \end{array}$$

The "abelian" part of $G_{\mathbb{Q}}$ is the (infinite) quotient of $G_{\mathbb{Q}}$ which corresponds to $\mathbb{Q}^{ab}$, the minimal subfield of $\overline{\mathbb{Q}}$ containing all finite extensions of $\mathbb{Q}$ with commutative Galois group. It is isomorphic to $\hat{\mathbb{Z}}^*$ (the multiplicative group of the profinite completion of the integers). Whilst the structure of $\mathbb{Q}^{ab}$ and the action of $\hat{\mathbb{Z}}^*$ on it are relatively well understood, we know very little about the subgroup $\Gamma = Gal(\overline{\mathbb{Q}}/\mathbb{Q}^{ab})$ of $G_{\mathbb{Q}}$, or about the way it combines with $\hat{\mathbb{Z}}^* = G_{\mathbb{Q}}/\Gamma$ to form $G_{\mathbb{Q}}$.

$$\begin{array}{ll} \overline{\mathbb{Q}} & \\ \mid \;\; \Gamma & \left.\right\} \\ \mathbb{Q}^{ab} & \quad G_{\mathbb{Q}} \\ \mid \;\; \hat{\mathbb{Z}}^* & \left.\right\} \\ \mathbb{Q} & \end{array}$$

Grothendieck's "dream", as he expressed it in *L'esquisse d'un programme*, is to understand combinatorial properties of $G_{\mathbb{Q}}$ by studying the way it acts on geometric objects, with the hope of obtaining a complete description of $G_{\mathbb{Q}}$. The idea is to reformulate geometric properties of certain varieties and in particular of certain moduli spaces, in terms of their fundamental groups and of the action of $G_{\mathbb{Q}}$ on these fundamental groups.

3. Geometric action of $G_{\mathbb{Q}}$

We are now going to describe some geometric groups on which $G_{\mathbb{Q}}$ acts. Let us consider Galois extensions of the field $\mathbb{C}(T)$ in one indeterminate over $\mathbb{C}$. There is a one-to-one correspondence between extensions of $\mathbb{C}(T)$ unramified outside of $0, 1$ and ∞ (meaning the ideals generated by the polynomials T, $T-1$, and the rational function $\frac{1}{T}$), and unramified coverings of the projective line with three points removed $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$: the field extensions are the function fields of the coverings

The Galois groups of the field extensions are the automorphism groups of the corresponding coverings; they are themselves quotients of the fundamental group of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$, which is the free group on two generators F_2.

We now consider unramified coverings of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$. Belyi's theorem asserts that if we have a covering $\pi : X \to \mathbb{P}^1_{\mathbb{C}}$, where X is a Riemann surface (i.e. a curve defined over $\mathbb{C}$)

such that the critical values of π lie in $\overline{\mathbb{Q}}$, then there is an equation for X with coefficients in $\overline{\mathbb{Q}}$ and a new rational function $\tilde{\pi}$ on X with coefficients in $\overline{\mathbb{Q}}$ and critical values in $\{0,1,\infty\}$.

Let $\pi : X \to \mathbb{P}^1_{\mathbb{C}}$ be a Belyi cover, i.e. X is a Riemann surface and the critical values of π lie in $\{0,1,\infty\}$. We can consider the field $\widetilde{\mathbb{C}(X)}$, compositum of the function fields $\mathbb{C}(Y)$ for all finite coverings Y of X. The Galois group $Gal(\widetilde{\mathbb{C}(X)}/\mathbb{C}(X))$ is the profinite completion $\hat{\pi}_1(X)$ of the fundamental group of X (i.e. the projective limit of the groups $\pi_1(X)/N$ for normal subgroups $N \subset \pi_1(X)$ of finite index).

Since X is defined over $\overline{\mathbb{Q}}$, it is in fact defined over a number field K (i.e. a finite extension of $\mathbb{Q}$). We have $\overline{K} = \overline{\mathbb{Q}}$, and every finite cover Y of X is defined over a finite extension of K. Let $K(X)$ denote the field of functions on X defined over K, $\overline{\mathbb{Q}}(X)$ the field of functions of X defined over $\overline{\mathbb{Q}}$, and for every finite cover Y of X, let $\overline{\mathbb{Q}}(Y)$ denote the field of functions on Y defined over $\overline{\mathbb{Q}}$. Let $\widetilde{\overline{\mathbb{Q}}(X)}$ denote the compositum of the fields $\overline{\mathbb{Q}}(Y)$ for all finite covers Y of X. A classical theorem states that $\mathbb{C}(X) = \overline{\mathbb{Q}}(X) \otimes \mathbb{C}$ and $\widetilde{\mathbb{C}(X)} = \widetilde{\overline{\mathbb{Q}}(X)} \otimes \mathbb{C}$, so the Galois group $Gal(\widetilde{\overline{\mathbb{Q}}(X)}/\overline{\mathbb{Q}}(X))$ is isomorphic to $Gal(\widetilde{\mathbb{C}(X)}/\mathbb{C}(X))$ which is isomorphic, as we noted above, to the profinite fundamental group $\hat{\pi}_1(X)$. However now the field $\overline{\mathbb{Q}}(X)$ is a field extension of $K(X)$ with Galois group $G_K = Gal(\overline{\mathbb{Q}}/K)$, and we have the following diagram :

$$\left.\begin{array}{cl} \widetilde{\overline{\mathbb{Q}}(X)} & \\ \mid & \hat{\pi}_1(X) \\ \overline{\mathbb{Q}}(X) & \\ \mid & G_K \\ K(X) & \end{array}\right\} \mathcal{G}$$

A theorem states that the huge group $\mathcal{G}$ can be written as a semi-direct product of $\hat{\pi}_1(X)$ with G_K. In particular, this means that there is an action of G_K on $\hat{\pi}_1(X)$ (in fact there are many such actions). In the case where X is defined over $\mathbb{Q}$, we now have an action of $G_{\mathbb{Q}}$ on $\hat{\pi}_1(X)$.

4. The moduli space $\mathcal{M}_{0,4}$

The moduli space $\mathcal{M}_{0,4}$ of Riemann surfaces of genus 0 with four marked points is isomorphic to $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$, which is a variety defined over $\mathbb{Q}$. So when we look at coverings of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$, we actually have coverings of the moduli space, and the arguments above show that there is an action of $G_{\mathbb{Q}}$ on $\hat{\pi}_1(\mathcal{M}_{0,4})$. This is very interesting, because mathematicians from other areas, ranging from mathematical physics to geometry, have studied such moduli spaces, and described many of their properties.

There is a very large group called $\widehat{GT}$ which also acts on the profinite completion of the fundamental group of the moduli space. Drinfel'd has given an explicit description of $\widehat{GT}$ as a set of elements satisfying some combinatorial conditions, so that we can actually make certain computations with elements of $\widehat{GT}$. Drinfel'd and Ihara have shown that $G_{\mathbb{Q}}$ is contained in $\widehat{GT}$. One would now like to know whether $G_{\mathbb{Q}}$ is all of $\widehat{GT}$, or how to identify $G_{\mathbb{Q}}$ as a subgroup of $\widehat{GT}$ so as to obtain an explicit description of its action on moduli spaces, and thus get a better grasp of the combinatorial properties of $G_{\mathbb{Q}}$.

The philosophy of Grothendieck's approach expresses the idea that the moduli spaces $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points (or "punctures") are the varieties defined over $\mathbb{Q}$ which contain *all* information about all the curves defined over $\mathbb{C}$. In particular they contain the information about the curves defined over $\overline{\mathbb{Q}}$, which in turn give us informations about $G_{\mathbb{Q}}$. Indeed, $G_{\mathbb{Q}}$ acts on all the profinite fundamental groups of the moduli spaces $\mathcal{M}_{g,n}$, respecting many natural morphisms between these groups which come from geometric morphisms between the spaces. The following questions are therefore natural ones to ask when we try to reach a further understanding of $G_{\mathbb{Q}}$ and of its relation to $\widehat{GT}$:

- What is the full group of ∞-tuples of automorphisms $(\phi_{g,n})_{g,n}$ of the $\hat{\pi}_1(\mathcal{M}_{g,n})$ for varying g and n, respecting all the natural morphisms between these groups? (It would seem that the answer is $\widehat{GT}$, as Drinfel'd suggests).
- The group of pairs of automorphisms of the "Teichmüller Tower" consisting of the two moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$, with natural morphisms between them, is precisely $\widehat{GT}$. To what extent do $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ contain "all the information" about all the $\mathcal{M}_{g,n}$?
- Since the moduli spaces contain "all the information" about a class of objects defined over $\overline{\mathbb{Q}}$, is it reasonable to think that no greater group than $G_{\mathbb{Q}}$, which is after all a purely arithmetic group, can act on the tower of their profinite fundamental groups?

Short Talks

Proc. of the 7^{th} EWM meeting, Madrid, 1995

The multidimensional Riemann–Hilbert problem, generalized Knizhnik–Zamolodchikov equations and applications

Valentina A. Golubeva

Let $L = \bigcup_{i=1}^{m} L_i$ be a reducible algebraic variety in $\mathbb{C}P^n$ and $\rho : \pi_1(\mathbb{C}^n \setminus L) \to \mathrm{GL}(n)$ a representation. The Pfaffian system of Fuchsian type with singular set L is the system of the form $dF = \Omega F$, where $\Omega = \sum_{i+1}^{m} A_i \frac{dL_i}{L_i}$, and $\sum_{i=1}^{m} A_i = 0$. The statement of the multidimensional Riemann–Hilbert problem is the following: for given ρ find the Pfaffian system of Fuchsian type whose fundamental solution realizes the given representation. For the simplest case of the variety L (L is the union of non–singular algebraic hypersurfaces with transversal intersections) some conditions of solvability were obtained (V.Golubeva, A.Bolibruch, T.Otsuki), some partial results are known in low dimensions also for more complex varieties L.

The Knizhnik–Zamolodchikov equations associated to the root systems A_n have their origin in the Wess–Zumino–Witten model of quantum field theory as the equations for the n–point correlation function. The construction of the different generalizations of these equations, in particular, for the other root systems B, C, D permits to consider these equations as examples of solvable cases of the Riemann–Hilbert problem. Indeed, for some root system R we are given the reducible algebraic variety in $\mathbb{C}P^n$, usually it is an arrangement of a finite number of hyperplanes, the known fundamental group of the complement to this variety in $\mathbb{C}P^n$ (the last results in this direction belong to D.Markushevich and A.Leibman) and a prescribed (by physical model) representation ρ. The generalized Knizhnik–Zamolodchikov equations for a variety of the assumptions on ρ was constructed by I.Cherednik, A.Matsuo, A.Leibman, V.Golubeva and V.Leksin, ect. The quantum variant of Knizhnik–Zamolodchikov equations (N.Reshetikhin and others) is known.

The generalized Knizhnik–Zamolodchikov equations have applications in contemporary quantum field theory and statistical mechanics: in the theory of quantum Hall effect, in the theory of anyons, super–conductivity, etc. The KZ theory is closely connected with the theory of many–body systems, described by the systems of Calogero–Moser–Sutherland type.

Quadratic and hermitian forms over rings

Laura Fainsilber

Université de Franche–Comte, Besançon, France
laura@math.univ-fcomte.fr

For quadratic forms over a field, Witt's cancellation theorem asserts that if q, q_1, and q_2 are three non-singular forms such that $q_1 \oplus q \simeq q_2 \oplus q$, then $q_1 \simeq q_2$. We want to generalize this cancellation to the situation of hermitian forms over rings. Let A be a ring with an anti-automorphism $\bar{\ }: A \to A$ of order 2, let M be a reflexive finitely generated left A-module. A hermitian form is a biadditive map $h : M \times M \to A$ such that $h(am, bn) = ah(m, n)\bar{b}$ and $h(n, m) = \overline{h(m, n)}$ for all $m, n \in M$ and $a, b \in A$. A form is said to be unimodular if the adjoint homomorphism it induces from M to its dual is bijective. If A is commutative and $\bar{\ }$ is the identity, then the hermitian forms are exactly the symmetric bilinear forms. We give counter-examples to show that the analog of Witt's cancellation theorem does not hold for symmetric bilinear forms over the ring of integers $\mathbb{Z}$. However, cancellation is possible for hermitian forms (unimodular or not) over rings which are finitely generated algebras over complete discrete valuation rings, such as rings of matrices over the p-adic integers $M_n(\mathbb{Z}_p)$ or group rings $\mathbb{Z}_p[G]$ for finite groups G.

Proc. of the 7^{th} EWM meeting, Madrid, 1995

On Sampling plans for inspection by variables

Vera I. Pagurova

Moscow State University, Russia

Designing of optimal sampling plans by variables is considered when the item quality characteristic follows a distribution belonging to a two parametric location–scale family of distributions. Two–class and three–class procedures of sampling plans by variables are investigated.

A random sample of n items is drawn from the lot. Inspection procedures are based on the measurement of an item quality characteristic X. An attributes plan bases the decision to accept or reject the lot only on the number of nonconforming items in the sample. Variables plans are able to achieve the same control with a smaller sample size by making use of the distribution of the initial measurements.

Let $X_1, X_2, \ldots, X_n$ be independent identity distributed variables with the common distribution density $f((x-a)/b)/b$, $|a| < \infty$, $b > 0$, a and b are unknown, $f(x) = F'(x)$, $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are order statistics, $\bar{X} = \sum_{i=1}^n X_i/n$, $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$.

1 – Two–class procedure. An item will be considered to be conforming if $X < u$, otherwise nonconforming. Set $p = \mathbf{P}\{X \ge u\}$. The hypothesis of interest is $H : p \le p_0$ ($100p_0$ is maximum allowable percent of nonconforming items) against $K : p > p_0$. We denote $d = (a-u)/b$, $d_0 = -F^{-1}(1-p_0)$, $Y_i = (X_i - a)/b$, $i = 1, \ldots, n$.

Theorem 1 *Let $f(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, then the uniformly most powerful invariant (u.m.p.i.) test at level α of H against K rejects H when $(\bar{X} - u)/S \ge c_\alpha$ with c_α defined by $\mathbf{P}\{(\bar{X} - u)/S \ge c_\alpha \mid d = d_0\} = \alpha$.*

Theorem 2 *Let $f(x) = \exp(-x)$, $x \ge 0$, then the u.m.p.i. test at level α of H against K rejects H when $(X_{(1)} - u)/(X - X_{(1)}) \ge c_\alpha$ with c_α defined by $\mathbf{P}\{(X_{(1)} - u)/(\bar{X} - X_{(1)}) \ge c_\alpha \mid d = d_0\} = \alpha$.*

Theorem 3 *Let $f(x) = 1$, $0 \le x \le 1$, then the u.m.p.i. test at level α of H against K rejects H when $(X_{(1)} - u)/(X_{(n)} - X_{(1)}) \ge c_\alpha$ with c_α defined by $\mathbf{P}\{(X_{(1)} - u)/(X_{(n)} - X_{(1)}) \ge c_\alpha \mid d = d_0\} = \alpha$.*

Now we consider an asymptotic approach. Denote $\mathbf{E}Y_1 = \mu$, $\mathbf{D}Y_1 = \sigma^2$, $m_j = \mathbf{E}((Y_1 - \mu)/\sigma)^j$, $\Delta(d)$ is some function of d, μ, σ, m_3, m_4; z_p is p–quantile of the standard normal distribution.

Proposition 4 *If $\mathbf{E}X_1^4 < \infty$, then the test of H against K with the rejection region $((\bar{X} - u)/S - (\mu + d_0)/\sigma)\sqrt{n}/\sqrt{\Delta(d_0)} \ge z_{1-\alpha}$ has the true level α when $n \to \infty$.*

An asymptotic test with estimators of a and b based on central order statistics is considered.

2 – Three–class procedure. An item will be considered to be conforming if $X < u_1$, to be marginally conforming if $u_1 \le X < u_2$ and to be nonconforming if $X \ge u_2$ where $u_1 < u_2$, $p_1 = \mathbf{P}\{X \ge u_1\}$, $p_2 = \mathbf{P}\{X \ge u_2\}$. The probability of acceptance of a lot of arbitrary quality (p_1, p_2) is studied.

References

[1] Lehmann, E.L. (1986) Testing Statistical Hypotheses. N.-Y., J. Wiley.

[2] Pagurova, V.I., Nesterova, S.A. (1991): Sampling Plans for Inspection by Variables. Moscow University. Comput.Math. and Cybern. 1.57 -63.

Evaluation

Evaluation of the mathematical aspects of the meeting

Isabel Labouriau

Universidade do Porto, Portugal

At the end of the meeting a discussion took place in order to evaluate the different mathematical parts. The following is a report of that discussion.

The general feeling was that the 3 series of talks and the interdisciplinary discussion on moduli spaces were of good standard. Non–trivial mathematics were presented in an understandable way, generated interdisciplinary discussions and even some informal workshops ("nothing to be ashamed of" was one comment). The talks had more participation than usual and speakers felt it was more interesting this way, that is, with a lot of questions asked, although this sometimes made it difficult for them to reach their goals.

The work of the "planted idiots" (members of the audience in charge of asking questions besides the spontaneous ones - a habit started at the EWM meeting in Luminy) was important to keep the talks at the right level. They created an informal atmosphere in the first talks that was kept through the end. In the last talks there were no "planted idiots"; the only questions were the spontaneous ones but it was pointed out that we should have kept the good habit to the end. For next time it was suggested that speakers get more information on the type of talk expected - some talks underwent big changes at the last minute.

It was agreed that three topics in mathematics plus one interdisciplinary discussion together with the non mathematical discussions was too much for one week. We should be less ambitious for the next meeting and have more time for talking about maths instead of listening. Too little time was allowed for the discussion on moduli spaces - the organizers were not sure how it would work - and it turned out to be so interesting it was continued in the spare time. One suggestion is that we have fewer talks towards the end of a meeting in order to have time for interdisciplinary discussions, like the one planned on moduli spaces and the spontaneous one on renormalization.

There were several opinions on the best way to organize a series of talks next time, opinions varied. The choice is not only between summer–school type or research –conference type talks; we may want to do something new, talks that generate an interdisciplinary discussion. For each series we may need an introductory session to introduce the language and basic results. This is hard on the first speaker and where do we stop when going backwards? The goal of the series is not to learn the language but to transfer ideas with a minimum of language. Two of the series in Madrid had an introduction that was necessary, but not the main goal. Some fields may be naturally more technical and need more introduction; some areas also have a tradition on non-technical talks. Maybe we should not spend too much time on elementary things, after all, one may understand a lecture without the details and appreciate it; we are used to that. With such a wide audience it is difficult to avoid some form of introduction, the question is how much.

The role of the organizer of a series was also discussed and how much she should interfere, either directing the speakers and choosing the topics so as to reach a certain goal or just trying to get the best maths we can, even at the cost of some coherence. Again some felt that it depended on the subject, as it is difficult to give a good idea of the field with homogeneous talks.

The talks that were not part of a series had a small audience - partly because the program was already heavy and partly for the lack of advertising, those interested could not always attend a talk. Parallel sessions were not thought to be a good solution, it makes talks more difficult to attend and somehow tend to become a secondary hierarchy without any scientific basis. Many of us prefer posters as they can be read at one's own speed and it is easier to ask questions. The problem is that people usually do not make good posters, maybe the first time we need guidelines. It might work if everybody had a poster, with a photo on it and maybe some historical background, so the posters would be less linked with hierarchy and take the place of introductions. It would be good to leave them on all the time and have a social event near them to break the ice.

OTHER TOPICS

In between meetings: the "everyday" life of EWM

Report by Sylvie Paycha

International EWM meetings are organised every two years; they offer an oportunity for women mathematicians from all over Europe to meet together, exchange ideas on mathematics as well as on women and mathematics. However, once every other year is seldom, and we feel EWM should go on "living" in between meetings. What could the "everyday life" of EWM be? E-mail offers many possibilities, such as sending the Newsletters to members of EWM and thereby keeping them informed of what is going on in Europe in the way of women and mathematics, discussing plans for the future among members of the standing committee, organising the proceedings of the last conference, ... However, exchanges via e-mail is a bit immaterial and maybe not fully satisfactory as the "everyday life" of an organisation!

We have thought of trying to organise concrete projects between the meetings of EWM which could be smaller scale meetings around a specific topic involving a few members of EWM interested in the topic.

A first attempt in that direction is an interdisciplinary workshop on "Renormalisation in Mathematics and Physics" (of which you will find an announcement here) which will take place in Paris. The idea of the topic of this workshop came up during the EWM meeting in Madrid where spontaneous discussions arose as to the different interpretations of the notion of renormalisation, a concept which was mentioned in various talks (in the field of statistical physics, of complex dynamical systems, of quantum field theory) in the course of the conference.

Such meetings between the general international meetings of EWM are a step towards a more concrete "everyday life" of the organisation!

Renormalisation in Mathematics and Physics

Paris, June 14 and 15, 1996

Preliminary announcement

The purpose of this workshop is to study the various interpretations of renormalisation in dynamical systems, statistical physics, and quantum field theory, and to explore the connections among them. Preliminary discussions on this subject took place at the last EWM meeting in Madrid in September 1995.

The workshop, a small scale two day meeting, will consist of four sessions, two each day, each session having at most two talks. Each session will include ample time for discussion by the participants of the various manifestations of renormalisation.

Abstracts of the talks should be available at the workshop and more detailed proceedings of the workshop, including the discussions and additional comments by the speakers, will be available afterwards.

Location. Institut Henri Poincaré, Paris, France (where *femmes et mathématiques* has its office). Accomodations will be organised depending on the number of participants (either in private homes or in a student hall).

Programme

Friday session
morning session: Renormalisation in dynamical systems(I)
speakers: Laura Tedeschini-Lalli (Rome), Betta Scoppola (Rome)
afternoon session: Renormalisation in dynamical systems (II)
speaker: Núria Fagella (Barcelona)

Saturday session
morning session:Renormalisation in statistical physics and quantum field theory
speaker:Annick Lesne (Paris)
afternoon session: Extension of the notion of renormalisation to infinite dimensional geometry
speaker: Sylvie Paycha (Clermont- Ferrand)

Other speakers will be announced later on

Scientific committee
Brodil Branner, Lyngby, Denmark
Laura Tedeschini-Lalli, Rome, Italy
Sylvie Paycha, Clermont-Ferrand, France

Organising committee
Colette Guillopé (Créteil, chairperson of *femmes et mathématiques*)
Sylvie Paycha (Clermont-Ferrand, convenor for EWM)

For further information, please contact Sylvie Paycha at:
paycha@ucfma.univ-bpclermont.fr
tel:(33) 73 40 74 42

Fe-mail/E-mail discussion

Report by Sylvie Paycha and Capi Corrales
Based on notes taken by Ute Bürger and Antje Petersen

The use of e-mail as a mean of communication between mathematicians is now wide spread around the world. As a new way of exchanging information and ideas, it can have some impact on how the mathematical community functions. We felt it was time for us to talk about the assets and draw-backs of such a mean of communication, and more specifically for women mathematicians.

Time for discussion was short and at this stage we can only formulate a few questions around this topic which we hope will lead to further discussions on and use of e(fe)-mail!

- **What is the status of the language used in an e-mail message?**
  It seems to be neither spoken nor written, but something in between. This "in between" status has the draw-back that it can lead to very abrupt messages (like "No!" as an answer to a question), if the person who sends the message does not feel the need to write a few words of introduction as one otherwise would in written form. In this sense it can sometimes be felt as an unpleasant (although fast) mean of communication. On the other hand, people sometimes feel free to suggest an idea in an e-mail message that is not yet "ripe" enough to be formulated in usual written form but which is still worth sending by e-mail. This leads to the second question:

- **Does "e-mail" modify the way mathematics are "done"?**
  New ideas can be exchanged rapidly by e-mail even if they are still in the process of being elaborated. They can therefore be elaborated in collaboration; a deep idea can come from many ideas (even if superficial) suggested by many people. The notion of "authorship" (which was and still is so important for recognition in the mathematical community) for one idea might gradually loose its meaning because of this superposition of ideas coming from different people. But what if one is not in a discussion or information network and cannot participate in this "common" elaboration of ideas ?

- **What about exclusion phenomena via e-mail ?**
  An e-mail discussion or information net grows fast but it can easily happen that one does not benefit from it by lack of information or because one does not have access to computers on which one can enter these nets. Since a lot of information mathematicians use nowadays transits through these means of communication, there is a clear segregation between mathematicians who use these nets and the others, who for some reason or other, are excluded from them.

- **What about women mathematicians and e-mail?**
  Which among the advantages/draw-backs mentioned above of this new mean of communication benefit to/disadvantage women mathematicians ? Do women mathematicians have in actual facts less access to it, do they use it less and why ? Some concrete suggestions came up during this informal discussion, such as

  – to organise a technical workshop during the next general EWM meeting to make women more familiar with the possibilities of the internet,

  – that Riitta Ulmanen could write a manual about how to make the best use of e-mail within EWM.

Mathematical studies at European universities

Magdalena Jaroszewska
Adam Mickiewsicz University, Poland

Data enclosed in this paper have come partly from information booklets and partly from direct communication with teachers and students of a number of European universities.

The European universities began to develop in the 12th century, the first ones stemming from monastery schools. In spite of numerous transformations they still preserve their traditional medieval organization and status, including divisions into faculties, structure of authorities, autonomous management. Some of the oldest universities established in 12th - 14th centuries are Bologna, Paris, Oxford, Cambridge, Parma, Salamanca, Valencia, Padoa, Naples, Rome, Prague, Cracow, Vienna, Heidelberg, Köln and Barcelona. The Sorbonne in Paris provided the university organizational pattern for other universities.

In our days there are numerous new scientific disciplines under development which increases demand for highly qualified specialists. The university graduate should have broad general education and also be well prepared for highly specialized professional career. How to balance the two aims, i.e. broad horizons and good professional training, in education of students? This goal poses still difficult problems for people and institutions responsible for education.

Quite recently, extensive political and social changes have been taking place in Europe. Students of today may choose place for studies practically all over Europe. Also the universities are changing in many ways, responding to market rules and economic pressures of the new Europe. The academic world tends to show a more elastic approach both in admission of students and in programs, offering the students the chance to choose more freely their own line of studies.

The system of university studies differs quite distinctly in different countries and even between universities in the same country.

Most often, the students start the studies at the age of 18 - 19 years. The studies are performed at most of the universities in three stages, corresponding to the degrees of *Bachelor of Science*, *Master of Science* and *Ph.D.*, respectively. The duration corresponding to these stages differ quite significantly.

The first stage of mathematical studies lasts two to four years during which the student acquires basic education. The principal subject, mathematics, is frequently accompanied by another one like physics, informatics, economy, psychology.

The second stage lasts one to three years. The studies are more specialized and the individual work of the students is emphasized. Most students end their formal education after this stage.

The third stage, Ph.D. studies, is undertaken in general by few students. Each of the students remain under care of a promotor of his/her choice and is given a subject for scientific work, usually within a specialized branch. As a rule, the students have to pass some exams at this stage, but their principal aim is to present a paper containing some new own results.

Let us look at some of the countries.

BELGIUM. Mathematical studies take 4 years and the first *diploma* - of the *Candidature* or *Baccalaureate* - is granted after the first two years. The diploma carries no practical significance. After 4 years of studies the student receives the *Licence Diploma* which is accepted at the labor market.

DENMARK. Mathematical studies are often combined with some other subject like informatics, physics, statistics. After 3 years the students receives B.Sci. degree and, after another two years the degree of *Candidatura Scientiarum*, corresponding to Master degree. A Ph.D. degree consists of a Master degree followed by a 3 year research training program. In Denmark, there are several universities with alternative organisation of studies. At Roskilde University, for example, all degree courses commence with a two year basic studies program. The basic studies program is a broad introduction to the humanities, the social sciences or the natural sciences.

ENGLAND. The first three years-long stage provides the student with the degree of the *Bachelor of Science*. The second stage of 1.5 – 2 years grants the student the title of a *Master of Science*. The third stage yields the title of a *Philosophy Doctor* (Ph.D.) and, in conjunction with the earlier stages, completes the 6 years studies. During the first three years frequently only one branch of science is taught while the remaining subjects are treated only marginally. Some universities, in particular the newer ones, offer a variety of *Single Honours*, *Joint Honours*, *Combined Honours*. For example, at the University of Birmingham the spectrum of 3–years long studies includes, among other, the following types: Single Honours in – mathematics, pure mathematics, applied mathematics, mathematics and statistics, statistics; Joint Honours in – mathematics and computer science, mathematics and psychology, mathematics and sport science; Combined Honours in – mathematics and ancient history, mathemetics and French studies, mathematics and music, etc.

FRANCE. The first stage of studies takes 2 years and provides the student with the *Diplome d'Etudes Universitaires Generales*. In the course of the second two years–long stage *La Licence* is granted after the third year and *La Maitrise* after the fourth year of studies. Ph.D. title can be obtained after the subject stage, which lasts 3 to 5 years.

FINLAND. After 5 years, the first diploma – *Filosofian Kandidaatti* and after the next 3 years – *Filosofian Tohtori* are awarded.

GERMANY. The studies last 4 to 5 years, they lack the first stage and yield no title, which would correspond to the B.Sci. title. After 4 – 5 years of studies one can get the title *Diplommathematiker(in)*. The curriculum of studies may include two branches of science, the principal one, e.g. mathematics, and the accessory one, e.g. chemistry.

HOLLAND. Similarly as in Germany, no title is given which would correspond to B.Sci.. The student is given the title of *Doctorandus*, an equivalent of M.Sci., after 4 years of studies and after the subsequent 4 years obtains Ph.D. title.

ITALY. All university studies take usually 4 years. After 4 years, the student receives the title of *Laurea Dottore* which corresponds, more or less, to the B.Sci. degree. The next stage lasts 3 to 5 subsequent years and yields the title of *Dottorato di Ricerche* (Ph.D.).

NORWAY. The title of *Candidates Magistrates* can be reached after 3.5 years of studies, *Candidates Scientiarum* after further 3 years and *Doctor Scientiarum* after about 3 more years.

POLAND. Until recently the uniform studies lasted 5 years. Now, Polish universities offer 3 to 3.5 years studies yielding the title of *Licentiate* with the possibility of prolonging them by another two or three years and obtaining the title of *Magister*. The curriculum of the 5 years long studies frequently contains the list of subjects required to obtain "in passing" the title of Licentiate. After 4 more years of studies the title of *doctor* can be reached.

PORTUGAL. Four years of studies lead to the *Licenciatura*. After additional 2 years of studies the student may obtain the title of *Mestre* and after another 3 to 4 years a Ph.D.degree.

SPAIN. The title *Licenciado* can be obtained after 5 years and *Doctorado* after 4 more years.

SWEDEN. The first 3 years of studies lead to the title of *Filosofie Kandidat*. After 1 – 2 more years of studies the degree of *Magister* and after subsequent 3 to 4 years – the Ph.D. degree can be reached.

SWITZERLAND. At some universities, the title *Diplomierte Mathematiker(in)* can be reached after 4 to 5 years of studies. After 4 more years, the title *Doctor Philosophiae* can be received.

By the years of studies we mean academic years. An academic year for example includes 30 weeks (Poland), 40 weeks (Sweden) or 42 weeks (Holland), organised in either two semesters or three trimesters. In most countries, the program corresponding to the M.Sci. degree includes 2.5 to 3 thousands hours of classes, lectures, laboratory exercises and seminars.

Some universities preserves traditional model of studies, in which the student is confronted with a curriculum, distributing all subjects to individual years with specified terms at which given credits should be obtained or exams passed. In many countries the point system of studies is being introduced. To be graduated the student has to collect an adequate number of credit points for obligatory subjects as well as for optional subjects. For example, at the University of Amsterdam (UVA) all parts of the 4–year program in M.Sci. studies form modules of the same size. For the course load of a given module the student receives 7 credit points. The course load is measured in hours: 1 point = 1 week of studies =5 days x 8 hours of work = 40 h of work (including about 20 h of classes + 20 h of individual work). One year of studies = 3 trimesters = 3 x 14 weeks of studies = 42 points. Then 4 years of studies = 168 points. Each subject can provide a defined number of points. At the first year of studies the so called propedeutic year, all subjects are obligatory. At the year 2, 3, 4 (the so called doctoraal phase) some subjects in individual specialities are obligatory and some are elective. The studies are highly individual, the students conciously shape theirs curriculum of studies.

Examination systems differ very strongly. Frequently, the students pass exams after finishing each course or during the year/semester at which lectures on the subject were given. In some countries the examination is held after two or more years of studies. In most of European countries the examination system oscillates between these two extremes. Let us screen the patterns at some of the countries resp. universities.

DENMARK. Each subject culminates in the form of an exam at the end of semester or academic year. Interestingly, the examining body includes the lecturer but also an additional professor.

SCOTLAND. Universities grant degrees with evaluation of the relevant qualifications by external examiners.

ENGLAND, Oxford. Students are examined at the end of the first and the third year of studies. Within a week the students pass 8 written exams, each lasting 3 hours. The exams test fluency in the material of obligatory subjects, both in the basic and in highly specialized branches of science.

GERMANY. The students have 2 main exams: *Vordiplom–Prufung* at the end the second year of studies and *Diplom–Prufung* at the end of the fifth year of studies.

PORTUGAL. Knowledge of most basic disciplines is tested by a single written test exam at the end of each semester. If the student fails at the exam - he/she can correct the result passing the oral exam.

According to the available information, the first stage of mathematical studies includes basic subjects common to majority of universities while curricula of mathematical studies differ a lot between universities at the second stage of the studies.

Almost all universities offer the following program:

- Mathematical analysis : sequences, limits, continuity, derivatives, indefinite and Riemann integrals of functions of one and several variables, curvilinear and surface integrals, ordinary differential equations.
- Linear algebra and algebra of basic algebraic structures.
- Euclidean and analytic geometry.
- Principles of informatics, numerical analysis, probability and statistics.

As evident from the above, European universities differ from each other in mathematical studies by the system of studies, the ways in which subjects are taught and the exams are conducted, they grant distinct titles and grades. Only the portion of basic mathematical knowledge is common to majority of studies curricula at the preliminary years of studies.

The following questions arise:

- Is it purposeful (and possible) to harmonize or standardize curricula of the first years of studies and to do the same with the granted degrees in the contemporary Europe with open borders and possibilities of free choice and change of place of studies ?
- Is it possible to define a certain European standard in the range ?

These problems have been discussed at the Round Table of European Congress of Mathematics [3].

Suggestions: The first step towards unification is comprehensive information. It would certainly be very useful if mathematical institutes/faculties could publish information booklets in English. The booklets should contain curriculum of mathematical studies. This would greatly facilitate work and decisions of both students in mathematics and the persons in the institutes/faculties responsible for education and for student transfers.

References

[1] European International Education Yearbook 1994, Nexus Business Communication, LTD, The annual to study abroad.

[2] Mathematics and Statistics, School of Mathematics and Statistics, The University of Birmingham, 1993.

[3] H.J. Munkholm, J.Netuka, V.Soucek, Degrees harmonization and student exchange programs, preliminary report for A Round Table at the European Congress of Mathematics, Paris, July 6-10, 1992.

[4] Notiziaro, Facolta di Science Matematiche Fisich e Naturali, Universite degli studi, Perugia, 1992/93.

[5] Programme des cours, Faculté des Sciences, Université de Mons Hainaut, 1992-93.

[6] Roskilde University, Denmark, Information Office, 1993.

[7] Stokholm Universitet, Information about students, 1993/94.

[8] Faculteit der Wiskunde en Informatica, Universitet van Amsterdam, Studiegids, 1992-93.

[9] Université de Paris-Sorbonne, (Paris VII), Livret de l'étudiant, 1993-94.

Family versus Career

The activity around the discussion on 'Family versus Career' was organized by science historian Eulalia Pérez Sedeño from Universidad Complutense de Madrid. In the spring of 1995 she formulated a questionnaire which was distributed by EWM. The paper which follows is her record of the 53 answers which she received.

Eulalia presented the results of her studies at the EWM meeting, following which a lively discussion took place. Participants emphasized that only a tiny amount of material was gathered, that no statistical conclusions could be drawn, and that those who answered mainly represented 'survivors' in the mathematical community.

We would like to make quite clear that opinions and conclusions expressed in this paper do not necessarily agree with those of EWM as an organization or those of its members.

Family versus Career in Women Mathematicians

Eulalia Pérez Sedeño
Universidad Complutense de Madrid, Spain

For a long time it was claimed that the fact that girls chose less scientific subjects than boys was reflected in the number of female scientists; and as the number of women studying sciences is smaller, smaller too is the number of women reaching professional success in that area. However, today we know that in some countries the global number of women in university is greater than that of men[1]; but there are fewer women studying physics or mathematics; the distribution of women in the different scientific specialities is unequal. When women surpass men, as well as when this is not the case, neither the quantity of professionals corresponds to the quantity of women prepared for this, nor is the distribution in the different knowledge areas equal; and there is no equality in the positions that men and women occupy[2]. Furthermore, a few sociometrical studies seem to show that the scientific careers developed by men and women are different (Cole [1979] and Zuckerman et al. [1991]). Why this is the case, is something sociologists and psychologists have attempted to explain in many various ways.

Many scholars tend to blame partnership and maternity for the fact that women do not reach, in great quantities, the working market in general and do not access the highest positions in all the scientific areas [3]. The underlying argument proceeds, more or less, in the following way: the pursuit of a scientific career is a full-time job; women carry the responsibilities of the household (they take care of the house, children, sick or old people, etc.). Performing the housewife functions takes a lot of time. For that reason, women either decide for the family (and thus there are less women than male in these professions) or their work is resented in terms of their scientific performance and efficiency. That would mean, for example, that single women would have to progress just as single males. In the same way, if it is said that marriage and children reduce the productivity of women, we would have to examine whether married women with children produce less than men in the same situation. There would have to be differences between married and unmarried women, among women with children and those without; and in other words, men and women with a similar quantity and quality of publications would have to have equal status. In fact, few empirical studies focus on these problems. The few existing studies deal with a few North American women belonging to various scientific disciplines [4], and they suggest the contrary (Zuckerman et al.[1991]). However, it is continuously asserted that the family is an obstacle to reaching full equality in science.

The present paper presents the results of a study accomplished through the association "European Women in Mathematics". A questionnaire was sent out through its network. It was intended to analyse educational, professional and economic status of the respondents and their households of origin; the development of their professional career - year of completion

[0]This research has been partially supported by Spanish DGICYT, project number PB92-0846-C06-02 and by the Researchers' Temporary Mobility Program funded by the Spanish Government.

[1]That is the case in Spain; see Pérez Sedeño [1996].

[2]See, for instance, Cole[1979], Rossiter[1982] and Pérez Sedeño[1996].

[3]On territorial and hierarchical discriminations see Rossiter[1982] and Pérez Sedeño[1995].

[4]See Zuckerman, Cole and Bruer [1991] and Cole [1979]. The study made by Jaiswal [1993] is more extensive so it is based in 158 women and 122 man. But it does not distinguish between different sciences and it is made in India, a country with a different structure.

of studies, first job, Ph.D. and publications, if any; the time they devote to domestic tasks, that is supposedly subtracted from study time and research, and it was intended to examine too, if differences in the previous factors result in different levels of performance; but also it claimed to grasp how these women perceive other women mathematicians, how they perceive themselves, what qualities they consider women mathematicians should have in order to perform their work, if they feel discriminated against, etc. In short, it was intended to throw light on some aspects of life of a handful of female mathematicians, so we could understand as much as possible about their situation and reality, with no vague speculations.

We claim to have obtained informative and qualitative data. We must take into account the fact that the sample is extremely small. Accordingly, anytime percentages are offered, they must be understood as merely informative of the situation, opinion etc. of the respondents. At no time is it intended to make any extrapolation nor generalization about the joint total of all the European women mathematicians. And moreover it would be ridiculous to do so. Furthermore, many of the analysed aspects are very difficult to quantify, just as what happens in the case of sex-based discrimination.

I also have to indicate that, from the point of view of social science, it is very important for the results that the interviewee does not know what working hypotheses guide the researcher or what he/she is looking for. Otherwise, their answers can be guided so they can think of answers that they would not have considered or they can conceal other ones. For that reason, I attempted to hide the pursued objective. In any case, the particular nature of the association, and the questionnaire didn't leave a very wide margin to speculation[5].

The questionnaire had both closed and open-ended questions and it was very long; it was claimed it was directed to women from different countries, at different ages, with different professional, economical and family situations; and even taking this into account, many questions had to be obliterated.

We do not know the exact number of women the questionnaire reached, but we estimate that it was transmitted to some two hundred women. 53 answers were received. The respondents belong to almost all the European countries. In this sense there are only two notable things: the absence of women mathematicians from the "eastern" countries and from Ireland, Switzerland, Austria and Greece; furthermore nine respondents live in the U.S. And in connection with mobility in work, only eight of the respondents work in a country which is different to their country of origin.

The occupations of these women are in the academic world, as I assumed originally, and 4 students answered too. Also there is one technical writer and one housewife. They describe their work in a variety of ways: sometimes they are very specific and sometimes they are very general and vague. The most prevalent activity is the teaching of mathematics (in compulsory - six to sixteen years old - or upper secondary schools - sixteen to eighteen - , for undergraduates, graduates or Ph.D. students); this is closely followed by research. But only 7 respondents say they are performing administrative tasks. All of them belong to several scientific societies, except 2, (the average is to belong to 3-4 societies or working groups).

These women were aged from 21 to 60 years old, and they were distributed in the following way: 12 women were aged from 51 to 60 years old (Group 1). The so-called Group 2 was formed by 10 women from 41 to 50 years old. The greatest number of women (Group 3) were from 31 to 40 years old (22 women, that is to say 41.5%); and only 8 women were

[5]But some surprising reactions were produced. Many respondents said their partners were mathematicians too, but a few said that they didn't believe their partners were interested in answering the questionaire (they were asked to pas the questionaire to their partners in the case were they were mathematicians too).

from 21 to 30 years old. In an academic context, and generally in the labouring world, the age is extremely relevant. With rare exceptions, there is a minimum age at which to begin university studies, to begin to work and a maximum age to retire. The age is also a frame of reference with respect to maturity and experience in work. Generally speaking, a greater service time gives as a result a promotion, a better position and better salaries. This remains fully confirmed in the case of our respondents, since the proportion of permanent positions diminishes as age decreases. But it must be emphasized that the global percentage of women that have obtained permanent or tenured positions in their workplace is very high: 70.58% [6].

The socioeconomic background of the respondents' results is relevant in order to understand their place in the social structure. It also must be taken into account that in traditional societies, individuals form a part of the family and this shapes them. Because of this, it does not appear strange to suppose that the fact that women become a member of professions dominated by men - and mathematics is one of them - is influenced by the sexual role, the socialization and the upbringing of girls in their families. The education and profession of the parents, as well as the family income facilitates or hinders the education and profession of women. Above all, the parents educational level is a very important factor in order to provide a better education and to select a suitable occupation.

The socioeconomic status of the original households of all these women was quite uniform; it was middle class, though 18% said they proceeded from a lower middle class home and 18% proceeded from a higher middle class household.

The educational status of fathers in Group 1 is quite uniform too: all of them possess a high educational level that moreover is superior to the mothers' one: 66.6% of fathers had university degrees, and a third of them had a Ph.D.; concerning the mothers, only 25% had a university degree, and another 25% had attended high school. The fathers' professions are varied, but 41.6% were university teachers; also there were attorneys, engineers, etc. Curiously, in this group only two mothers did not work; and the professions of the rest of the mothers were varied, although a third of them are teachers. Fathers from Group 2 had a high educational status: only Group 3 fathers did not possess university studies; half of the mothers did not have university degrees or similar education. In relation to fathers' professions the academic ones were numerous, and the rest of the fathers were business men, farmers or architects. Four mothers, out of the total (10), did not work; the professions of the others were: architects, psychoanalysts, business or mathematics teachers. In Group 3, just three fathers didn't have university education and seven fathers had a Ph.D.; engineers, architects and teachers with scientific training, and especially mathematical knowledge, were numerous. Regarding the mothers, 27.2% were housewives; the rest were distributed among different professions, but teaching was the most common (36%). Finally, parents in Group 4 were the most assorted economically, educationally and professionally. Although just 8 women make up this group, all the economic situations of the questionnaire appear; and in connection with the educational status, three fathers have university education and there is just one (mathematics) teacher, the rest of the professions varies. The educational status of the mothers in this group is slightly superior to the fathers' one: 5 obtained university degrees and two high school degrees. There are three teachers, two administrative officers, a secretary, a banker and only one housewife.

Marriage or partnership and the children, if any, and the time these women devote to the household are very important in this research, as we can learn how their family situation

[6] Men who answered the questionaire are not included here.

affects or has affected their professional performance or efficiency. In order to measure professional performance four factors have been used: the year of completion of the undergraduate degree, the year they obtained their Ph.D. degree, the year they began to work (remunerated work) and the number of publications [7].

Marriage or partnership[8] provides a new status to individuals. It usually confers new roles and positions: on one hand, society expects, as a minimum, that households as well as the upbringing of children and care of older family members, if any, shall be the domain of women; additionally, the position the household occupies in society tends to sum up the status of both members of the couple[9]. All women in Group 1 have or have had a partner. Generally they obtained a partner late, supposedly after having established their professional career, in a certain way. Just two of them obtained a partner before the age of 24 and just one was a student. All the partners of these women have obtained a university degree in science, for the most part mathematics or related subjects; we do not know about two of them since this information was not given to us. In Group 2, all women have or have had partners. Four women had their first partner when they were less than 25 years old. Mathematicians or persons with strong mathematical training, such as engineers, economists, and so on, are numerous among the partners of this group. In any case all of them have a university education, except one (where information was not given). Group 3 was composed of 22 women, and just five of them have not, or have not had, partners. Nine women obtained partners before they were 25 years old. All the women with partner, except two, had formal relationships with scientifically trained people (mathematicians, engineers, computer scientists, etc.). Finally, in Group 4 only four women have a partner: 2 of them obtained partner before the age of 25, the other two women later. Regarding the educational or professional status, three partners in Group 3 have scientific training and 1 is a travel agent.

Previous data show almost total congruency in educational and professional status among the couples. Such a fact supports the conjecture that, in our current society, such congruency is a very important variable in the vital situation of an individual. That congruency affects the lifestyle, the couples' behaviour and adjustment; it also provides stability and psychological reward. Nonetheless it is worth emphasizing that some women state that their partner has been one of the most principal obstacles they have found in the development of their career.

We can examine to what extent family situation affects the professional and scientific performance or efficiency in these women. In Group 1, women finished their graduate studies from 21 to 24 years old. All these women began to work from 20 to 24 years, except 2: one began when she was thirty years old (before she was married and had children) and the other when she was thirty seven years old, (17 years after her marriage and 11 after having her last child). Just two women did not have children and four had the first before they were thirty years old. That is to say, in this group 66.6% of women with children were "older" mothers, that is, when they were thirty years old or more. These children went to school when they were six or seven years old (just four children began school before).

However, neither partnership nor maternity have affected the performance or efficiency of these women, against what is many times claimed. Just one woman resigned from a tenured position to be devoted to her children, but the rest continued working after marriage and

[7] I am aware of problems implied by using these factors. For instance, some respondents have not finished their graduate studies yet; others spent so much time teaching that they can scarcely publish. And, concerning motivation, I am aware too that some questions would have to be asked.

[8] I am referring to heterosexual partnership; the homosexual one varies as it does social permissiveness.

[9] We must take into account that, in most countries, women loose their maiden name and they go on to be called by their husbands' names, that is to say, in a sense, their previous identity disappears.

children. Seven obtained their Ph.D. after having their first child and when at least one child was not going to school yet. Just in one case does the first childbirth and doctorate coincide. In relation to women with no children (just two), it does not seem they have had better performance than the other ones. Even though one of them obtained her Ph.D. and published her first paper by the time she was 25 years old, the other obtained her Ph.D. at twenty nine and published her first paper at thirty seven; this is not meaningful, since there are Ph.D. women at twenty four, twenty eight and twenty nine when they already had children. Concerning publications of women with children (and the subsequent pace of publication) they do not seem to be affected negatively by marriage or children: only one of them published for the first time when her youngest child was 11 years old.

In Group 2 all women finished their graduate studies between the ages 21 and 23: just one of them interrupted her university studies because of maternity and she finished some years later. They started to work at a slightly different age: just one at 23, seven from 24 to 28 and two when they were more than 30 years old. The seven women with a Ph.D. obtained the doctorate at similar ages: from 25 to 30. Although all had children much after their doctorate, their motherhood did not influence their scientific performance: all of them continued publishing at the same pace, and even to a greater pace, after having children. Just one published at forty two for the first time, much after her youngest child had begun to go to school. There does not seem to be a meaningful difference between those women and those who do not have children (three) in relation to their first publications: one of them started to publish at 26, one at 28 and the third at 40.

The trajectory in the university of women in Group 3 is standard too. Most of them started to work when they were between 24 and 30 years old, but one began at 22 and 3 from 30 to 33 years (and two of these do not have children). Thirteen women did not have children and twelve of them obtained their Ph.D. From nine women with children, eight have a Ph.D. ; and six of these mothers obtained their Ph.D. after having had their first baby and when at least one child was not going to school yet. The quantity and pace of publications does not vary because of marriage or maternity in most cases, as the average of publications of women with children is superior to that of women without children. Of course this must be taken with caution, since some respondents (4) have not answered this part of the questionnaire or they have answered partially. The analysis of the situation of the last group is not very informative, as no one has published, just two have obtained a Ph.D. and just one has a son.

The dedication to the tasks of the household varies among the people polled: from 28 hours per week that some women reveal until 0 hours that other ones assert. There is no difference between women living alone and women with a partner or family: 10 hours as a weekly average. Most interesting is how much time each member of the couple devotes to domestic work: in Group 1, five women say they spend the same amount of time as their partners, but three claim they spend more time than them on these tasks. In Group 2 just one spends the same amount of time as her partner, another one spends less time than her partner and in the rest of the cases they spend more time on household tasks than their partners. In Group 3, one woman asserts her partner spends more time on housework than she does, but the quantity of couples in which women do more household work than their partners is equal to couples in which the work is distributed in a similar way. Finally, in Group 4 all women answered they spent more time on household work than their partners.

All these women work in a masculine context; that is to say, in their workplace more men than women work and those male colleagues exercise control over greater number of persons (usually students). Mainly, they feel they are equally treated to their male colleagues in terms of salary and promotion (67 %) and responsibilities (60 %). But five women have a higher qualification, needed in order to perform their work and they could occupy superior positions, that is to say, they are underemployed. And the number of women that feel undervalued is very similar to those which feel themselves to be equally valued. In relation to the rest of the women, 87.1 % of the respondents think there are more women undervalued than men and 73.2 % of the respondents think that there are more overvalued men than women[10].

We can classify the support and the opportunities they have found in their career in two types: personal/intellectual and economic. In the first kind it should be emphasized that twenty respondents admitted having had a male counsellor, 2 a female counsellor and one respondent had both male and female counsellors. And some of the answers were "Meet interesting people", "enjoy mathematics", "invitation to collaborate on a problem", "support from colleagues and husband", "general interest", "ability to talk to others", "meeting people who have encouraged me", "many excellent programs to develop teaching methods". In relation to the second kind, scholarships or grants are the most cited items. The fact that their advisor helped them to find their first job, "opportunity to work as an assistant", "sabbatical periods", etc. are quoted too. It must be emphasized that only two women refer to programs for promoting women.

The answers concerning the obstacles they met in their careers are also varied, but family is the most quoted. When they say "family" they mean balancing between family and profession or to have to decide between family and career. And obstacles directly related to their sex are numerous: "Sexual and sex-based harassment", "most of the time the fact I am a woman", "some male colleagues resent my efforts to hire and retain women in the faculty", "bad rumours", "men were preferred in positions", "family problems", "partner's competitive relation to my career", "women who discourage other women, having to justify how much I contributed in joint publications". "Very rare opportunities for promotion" and "blatant discrimination" can be understood as sex-related too. However, there are many more: lack of people to talk to for research, loneliness, isolation in my department, ("which it is not deliberated to you by the men", one adds), incompetent advisors or lack of strong counsellors, lack of support from pure mathematics colleagues in the department, academic attitude towards teaching versus research, teaching looked down upon, time, not available positions, political corruption, favoritism and incompetent decisions at departmental level, and so on[11].

These few pages express a brief summary of the results of the questionnaire. I am aware of many things that are obliterated, but other aspects have come to the light. Some of them can be object of reflection by the association itself[12]. But many can be a matter of general thinking.

Social scientists don't agree on a unique definition of discrimination. The variance in their definitions reveals the different points of view in discriminatory behaviour scholars (Cole [1979]). And on sex discrimination the differences are great. The traditional concept of women, the socialization process of the girls and the social accepted position of women in family life seem indicate that women performance would have to go in a certain direction. But the results of the questionnaire show that pregnancies, upbringing babies, publications, Ph.D., positions, and so on are interwoven. And the result of the questionnaire says too that there are not meaningful differences with respect to the age at which single and married women, women

[10] Just one respondent said that there are more overvalued women.

[11] Many of these obstacles could be adduced by other academicians or scientists.

[12] For instance, there are few young female mathematicians. These can be because the foundational characteristics of the association. But it can be because the educational status of the household origin.

with children and without, obtain their Ph.D. and their positions, or when they publish. Usually these women have to adjust their career and their family, and, nevertheless, they obtain their Ph.D., positions and publish. So we can conclude that the unequal participation of women in mathematics has to proceed from cultural and social structured norms and values, not because they decide for family nor by their intellectual and academic achievements. I know that many of us presumed this assertion. But now, we have these data on our hands. And perhaps this paper can illustrate how science works as a social system and how science rewards women's participation in a male-dominated world. Perhaps we can learn from it that values and interests form part of science too.

Cole, J. R. [1979]: Fair science. Women in the scientific community, New York, The Free Press.

Jaiswal, R.P. [1993]: Professional Status of Women, Jaipur/New Delhi, Rawat Publications.

Oakes, J. [1990] Lost talent: the underparticipation of women, minorities and disabled persons in science, Santa Monica, Ca., Rand.

Ogilvie, M.B. [1986]: Women in Science. Antiquity through the Nineteenth Century. Cambridge, Mass./Londres. The M.I.T. Press.

Osen, L. [1974]: Women in Mathematics, Cambridge, MA., MIT Press.

Pérez Sedeño, E. [1996]: "Scientific Academic Careers of Women in Spain: History and Facts", in press.

Ramaley, J.A. (ed.) [1978] : Cover discrimination and women in the sciences, AAAS selected symposium, 14. Westview Press.

Rossiter, M. [1978]: "Sexual Segregation in the Sciences: Some Data and a Model", Signs, vol. 4, n 1. - [1982]: Women Scientists in America. Struggles and Strategies to 1940, The John Hopkins University Press.

Zuckerman, H., Cole, J. and Bruer, J.T. [1991]: The outer circle, New Haven, Yale Univ. Press.

Eulalia Pérez Sedeño
Universidad Complutense – 28040, Madrid
Spain
eulalia@eucmax.sim.ucm.es

COMMITTEES, COORDINATORS AND PARTICIPANTS

Committee Members

1995-1997

STANDING COMMITTEE:

– Sylvie Paycha (convenor) - e-mail: paycha@ucfma.univ-bpclermont.fr
– Polyna Agranovich - e-mail: agranovich@ilt.kharkov.ua
– Valentina Barucci - e-mail: barucci@gpxrme.sci.uniroma1.it
– Bodil Branner - e-mail: branner@mat.dtu.dk
– Capi Corrales (deputy convenor) - e-mail: capi@sunal1.mat.ucm.es
– Marie Demlova - e-mail: demlova@math.feld.cvut.cz
– Laura Fainsilber - e-mail: laura@math.univ-fcomte.fr
– Emilia Mezzetti - e-mail: mezzette@univ.trieste.it
– Rosa Maria Miró–Roig - e-mail: miro@cerber.ub.es
– Marjatta Näätänen (treasurer) - e-mail: mnaatanen@cc.helsinki.fi
– Ragni Piene - e-mail: ragnip@math.uio.no
– Caroline Series - e-mail: cms@maths.warwick.ac.uk.

Names of women who might be nominated for prizes should be sent to:

– Magdalena Jaroszewska - e-mail: magjar@math.amu.edu.pl

Link with Association for Women in Mathematics (AWM):

– Caroline Series - e-mail: cms@maths.warwick.ac.uk

Link with European Mathematical Society (EMS):

– Bodil Branner - e-mail: branner@mat.dtu.dk

Statistics about women mathematicians on different European countries:

– Kari Hag - e-mail: kari@imf.unit.no

e-mail EWM network:

–to subscribe contact Sarah Rees - e-mail: sarah.rees@newcastle.ac.uk

Coordinators

1995-1997

1. International Coordinators

East

Inna Berezowskaya
Center for Forest Ecology RAS
Arh. Vlasova st. 51
Moscow 117393, Russia

Marie Demlova
Department of Mathematics
Faculty of Electrical Engineering
Czech Technical University
Technicka 2
16627 Praha 2, Czech
e-mail demlova@math.feld.cvut.cz

Central

Marketa Novak
Department of Computer Sciences
Chalmers University of Technology
S-41296 Göteborg, Sweden
e-mail novak@cs.chalmers.se

West

Capi Corrales
Departemento de Algebra
Facultad de Matematicas
Universidad Complutense de Madrid
E-28040 Madrid, Spain
e-mail capi@sunal1.mat.ucm.es

2. Regional Coordinators

Belgium

Isabel Pays
Universite de Mons-Hairaut
Section de Mathematiques
15, Avenue Maistriau
B-7000 Mons, Belgium
e-mail spays@bmsuem11.binet

Czech

Marie Demlova
Department of Mathematics
Faculty of Electrical Engineering
Czech Technical University
Technicka 2
16627 Praha 2, Czech
e-mail demlova@math.feld.cvut.cz

Denmark

Bodil Branner
Department of Mathematics
Building 303
The Technical University of Denmark
DK-2800 Lyngby, Denmark
e-mail branner@mat.dtu.dk

Estonia

Helle Hein
Institute of Applied Mathematics
University of Tartu
2 Liivi Str
EE-2400 Tartu, Estonia
e-mail helle_h@vask.ut.ee

Finland

Marja Kankaanrinta
Department of Mathematics
P.O.Box 4 (Hallituskatu 15)
FIN-00014 University of Helsinki, Finland
e-mail mkankaanrint@hylk.helsinki.fi

France

Colette Guillopé
Universite Paris Sud
Bat 425
F-91405 Orsay Cedex, France
e-mail guillope@math.u-psud.fr

Germany

Christine Bessenrodt
Institut fuer Algebra und Geometrie
Fakultät für Mathematik
Otto-von-Guericke-Universität
PSF 4120
D-39016 Magdeburg, Germany
e-mail Christine.Bessenrodt@Mathematik.
Uni-Magdeburg.DE

Great Britain
Amanda Chetwynd
Mathematics Department
Lancaster University U.K.
Lancaster LA1 4YF, Great Britain
e-mail maa003@central1.lancaster.ac.uk

Greece
Vassiliki Farmaki
Department of Mathematics
University of Athens
Panepistimiopolis, 15784 Athens, Greece
e-mail sma20@grathun1.bitnet

Maria Leftaki
Department of Mathematics
University of Patras
26110 Patras, Greece
e-mail leftaki@math.upatras.gr

Italy
Emilia Mezzetti
Dipartimento di Scienze Matematiche
Universita di Trieste
Piazzale Europa 1
I-34127 Trieste, Italia
e-mail mezzette@univ.trieste.it

Latvia
Daina Taimina
The University of Latvia
Rainis Blvd 19
Riga, LV1586, Latvia
e-mail dtaimina@lanet.lv

Lithuania
Nerute Kligiene
A. Gostauto 12
2600 Vilnius, Lithuania
Institute of Mathematics and Informatics
e-mail nerute.kligiene@mji.lt

The Netherlands
Coby Geijsel
Fac. der Wiskunde en Informatica
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam, Netherlands
e-mail coby@fwi.uva.nl

Norway
Ragni Piene
Department of Mathematics
University of Oslo
P.O.Box 1053 - Blindern
N-0316 Oslo 3, Norway
e-mail ragnip@math.uio.no

Poland
Magdalena Jaroszewska
Faculty of Mathematics
and Computer Science
Adam Mickiewicz University
Matejki 48/49
60-769 Poznan, Poland
e-mail magjar@math.amu.edu.pl

Agata Pilitowska
Warsaw Technical University
Mathematics Institute
Plac Politechni 1
00-661 Warsaw, Poland
e-mail agapil@plwatu21.bitnet

Portugal
Isabel Labouriau
Grupo de Matematica Aplicada
FCUP
Rua das Taipas, 135
4050 Porto, Portugal
e-mail islabour@ncc.up.pt

Romania
Rodica Tomescu
Str. Transilvania Nr 52
Sect 1 70778 Bucuresti, Romania
e-mail tomescu@vala.elia.pub.ro

Russia
Inna Berezowskaya
Center for Forest Ecology RAS
Arh. Vlasova st. 51
Moscow 117393, Russia

Inna Yemelyanova
23 Gagarin Ave.
Nizhny Novgorod 603600
GSP-20, Russia
e-mail yemel@nnucnit.unn.ac.ru

Spain
Capi Corrales
Departemento de Algebra
Facultad de Matematicas
Universidad Complutense de Madrid
E-28040 Madrid, Spain
e-mail capi@sunal1.math.ucm.es

Sweden
Marketa Novak
Department of Computer Sciences
Chalmers University of Technology
S-41296 Göteborg, Sweden
e-mail novak@cs.chalmers.se

Switzerland
Francine Meylan
2-4 rue du lievre
Case postale 240
CH-1211 Geneve 24, Switzerland
e-mail francine.meylan@ima.unil.ch

Turkey
Betul Tanbay
Department of Mathematics
Bogazici University
Istanbul, Turkey
e-mail tanbay@boun.edu.tr

Ukraine
Polyna Agranovich
47 Lenin Avenue
Kharkov 164
310164 Ukraina
e-mail agranovich@ilt.kharkov.ua

List of Participants in EWM95

Denmark

1. Bodil Branner
Department of Mathematics
The Technical University of Denmark
Building 303
DK-2800 Lyngby
email: `branner@mat.dtu.dk`

2. Pia Willumsen
Institut des Hautes Études Scientifiques
F-91440 Bure-sur-Yvette, France
email: `pia@mat.dtu.dk`

England

3. Marjorie Batchelor
Department of Pure Mathematics
and Mathematical Statistics
16 Mill Lane
Cambridge CB2 1SB

4. Catherine Hobbs
School of Computing and
mathematical sciences
Oxford Brooks University
Gipsy Lane
Headington, Oxford, OX3 0PB
email: `cahobbs@brookes.ac.uk`

5. Alice Rogers
Department of Mathematis
King's College
Strand, London WC2R 2LS
email: `f.a.rogers@kcl.ac.uk`

Finland

6. Marjatta Näätänen
Department of Mathematics
PO Box 4 (Yliopistonkatu 5)
FIN-00014 University of Helsinki
email: `mnaatanen@cc.helsinki.fi`

7. Marja Kankaanrinta
Department of Mathematics
PO Box 4 (Yliopistonkatu 5)
FIN-00014 University of Helsinki
email: `mkankaanrint@cc.helsinki.fi`

8. Riitta Ulmanen
Secretary, Officce of EWM
Department of Mathematics
PO Box 4 (Yliopistonkatu 5)
FIN-00014 University of Helsinki
email: `ulmanen@sophie.helsinki.fi`

France

9. Laura Fainsilber
Laboratoire de Mathématiques
Université de Franche-Comté
16, route de Gray
F-25030 Besançon
email: `laura@math.univ-fcomte.fr`

10. Flora Koukiou
Université de Cergy–Pontoise
Department of Theoretical Physics
Cergy–Pontoise
email: `koukiou@u-cergy.fr`

11. Tan Lei
Mathematics Institute
University of Warwick
Coventry CV4 7AL
email: `tanlei@maths.warwick.ac.uk`

12. Mireille Martin-Deschamps
Departement de Mathématiques
Ecole Normale superieure
45 rue d'Ulm
F-75230 Paris Cedex 05
e-mail: `deschamp@dmi.ens.fr`

13. Sylvie Paycha
Department de Mathématiques Apliquées
Université Blaise Pascal
F-63177 Aubiere Cedex
e-mail: `paycha@ucfma.univ-bpclermont.fr`

14. Marie-Françoise Coste Roy
Université de Rennes 1
14 Rue de la Rabine
F-35510 Cesson-Sevigne
email: `costeroy@univ-rennes.fr`

15. Claire Voisin
Université de Paris XI
Département de Mathématiques
F-91405 Orsay
email: voisin@diam1.ens-cachan.fr

Germany

16. Ute Bürger
Sielwall 66
D-28203 Bremen
email: a05r@hatnix.statistik.
uni-bremen.de

17. Marlen Fritzsche
Universität Potsdam
Institut für Mathematik
PF 60 15 53
D-14414 Potsdam
email: mfritz@rz.uni-potsdam.de

18. Priska Jahnke
Salegrund 21,
D-35041 Marburg
email: radloff@mailer.uni-marburg.de,

19. Mara D. Neusel
Institut für Algebra und Geometrie
Fakultät für Mathematik
Otto-von Guericke-Universität
D-39016 Magdeburg
email: mara.neusel@mathematik
.uni-magdeburg.de

20. Antje Petersen
Gr. Johannisstraße 106,
D-28199 Bremen
email: antje@mathematik.uni-bremen.de

21. Christina Schmerling
Mauerstraße 7
D-28217 Bremen
email: tina@mathematik.uni-bremen.de

22. Sandra Hayes
Mathematisches Institut
Technische Universität
Arcisstraße 2
D-80333 München
email: hayes@mathematik.tu-muenchen.de

Italy

23. Valentina Barucci
Dipartimento di Matematica G. Castelnuovo
Universitá La Sapienza
Piazzale Aldo Moro 2
I-00185 Roma
email: barucci@gpxrme.sci.uniroma1.it

24. Marina Bertolini
Dipartimento di Matematica F. Enriques
via C. Saldini 50
I-20133 Milano
email: bertoli@vmimat.mat.unimi.it

25. Emilia Mezzetti
Department of Mathematics
University of Irieste
Piazzale Europa 1
I-34100 Trieste
e-mail mezzette@univ.trieste.it

26. Enza Orlandi
Dipartimento di Matematica
Terza Universitá
Via Segre 2
I-00146 Roma
email: orlandi@matrm3.mat.uniroma3.it

27. Laura Tedeschini-Lalli
Dipartimento di Matematica
III Universitá di Roma
Via Corrado Segre 2
I-00146 Roma
email: tedeschini@axcasp.caspur.it

28. Cristina Turrini
Dipartimento di Matematica F. Enriques
via C. Saldini 50
I-20133 Milano
email: turrini@vmimat.mat.unimi.it

Norway

29. Ragni Piene
Department of Mathematics
University of Oslo
P.O. Box 1053 Blindern
N-0316 Oslo
email: ragnip@math.uio.no

Poland

30. Magdalena Jaroszewska
Faculty of Mathematics and Computer Science
Adam Mickiewsicz University
Matejki 48/49
60-769 Poznan
email: magjar@math.amu.edu.pl

Portugal

31. Isabel Labouriau
Grupo de Matemática Aplicada, FCUP
Rua das Taipas, 135
4050 Porto
email: islabour@ncc.up.pt

32. Margarida Mendes Lopes
Depto. de Matemática
Facultade de Ciencias de Lisboa
R. Ernesto de Vasconcelos
1700 Lisboa
e-mail: mmlopes@ptmat.lmc.fc.ul.pt

Romania

33. Daniela Florenta Ijacu
Bd. Cuza Voda 150 apt. 24 Sect.4
7000 Bucuresti

34. Rodica Tomescu
Transilvaniei 52
70778 Bucarest
email: tomescu@vala.elia.pub.ro

Russia

35. Valentina Goloubeva
20A Usiyevich str.
Department of Mathematics
All-Russian Institut of Scientific and Technical Information (VINITI)
123219 Moscow

36. Vera Pagurova
Department of Computational Mathematics and Cybernetics
Moscow University
Vorobievy Gory
119899 Moscow

37. Elvira Perekhotseva
Hydrometeorological centre of Russia
email: kohan@glas.ape.org

Spain

38. Mariemi Alonso
Departamento de Álgebra
Facultad de matemáticas
Universidad Complutense de Madrid
E-28040 Madrid
email: alonso@sunal1.mat.ucm.es

39. Capi Corrales Rodrigáñez
Departamento de Álgebra
Facultad de Matemáticas
Universidad Complutense de Madrid
E-28040 Madrid
email: capi@sunal1.mat.ucm.es

40. Núria Fagella
Centre de Recerca Matemática
Universidad Autónoma de Barcelona
Apartat 50
E-08193 Bellaterra, Barcelona
email: nuria@manwe.mat.uab.es

41. Raquel Mallavibarrena
Departamento de Álgebra
Facultad de Matemáticas
Universidad Complutense de Madrid
E-28040 Madrid

42. Rosa María Miró-Roig
Facultad de Matemátiques
Departament d'àlgebra y geometria
Universitat de Barcelona
Gran Via de les Corts Catalanes 585
E-08007 Barcelona
e-mail miro@cerber.ub.es

43. Isabel Paniagua García-Calderón
Departamento de matemática aplicada
Facultad de matemáticas
Universidad Complutense de Madrid
E-28040 Madrid
e-mail: Paniagua@mat.ucm.es

44. María Victoria Rodríguez Uría
Universidad de Oviedo
Departamento de matemáticas
Facultad de Económicas
Oviedo

Sweden

45. Charlotte Lundvall
c/o Falck
Brahevägen 3
S-102 74 Stocksund
email: `f90-clu@nada.kth.se`

46. Marketa Novak
Departent of Mathematics
Chalmers University of Technology
S-41296 Göteborg
email: `novak@cs.chalmers.se`

Switzerland

47. Karin Baur
Imfeldstrasse 70
CH-8037 Zürich
email: `mikado@amath.unizh.ch`