Old and New themes in Number Theory

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Introduction

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DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis
Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM

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studied by Euler, Legendre, Abel, Jacobi...

Euler

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Abel

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where $f(x) \in \mathbb{Q}[x]$ is a cubic with distict roots;

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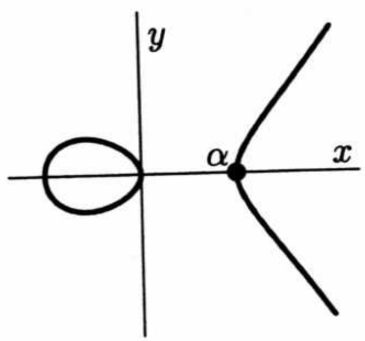
Viewed as a plane curve, its set of real points looks like



A Cubic Curve with Two Real Components

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Its solution set $E(\mathbb{Q})$, consisting of pairs (x, y) with x, y in $\mathbb{Q}$, together with the point at ∞ has an abelian group structure. More generally true that the set of F-rational points, $E(F)$, is an abelian group for any field F.

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Problems related to elliptic curves were however studied in a different context from around the 10th century as we shall see below.

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$g(E/F)$=rank of E/F, an arithmetic invariant.

A deep conjecture due to Birch and Swinnerton-Dyer (BSD) predicts that $g(E/F)$ is equal to the order of a conjecturally analytic function $L(E/F, s)$ at $s = 1$.

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Iwasawa theory uses p-adic techniques to bring together the following three different strands of Mathematics with elliptic curves occurring as a common motif:

- (i) Age old arithmetic problems
- (ii) special values of complex zeta and L-functions
- (iii) Algebraic questions concerned with study of modules

over Iwasawa algebras of compact p-adic Lie groups

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Important milestone: Work of Andrew Wiles (mid 1990's) and others, leading to the proof of modularity of elliptic curves over $\mathbb{Q}$ and consequently a proof of Fermat's last theorem.

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Definition: An integer $N > 1$ is congruent if N is the area of a right angled triangle, all of whose sides have rational length.

Example: 5,6,7,13,14,15,21,22,23,29,30,31,34..... are congruent numbers

- 5 = Area of right angled triangle with sides (9/6,40/6,41/6)
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Surprisingly, this is really a problem about *elliptic curves*

• Congruent number problem for an integer $N > 1$ leads very naturally to studying elliptic curves of the form $E : y^2 = x^3 - N^2 x.$

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Lemma: The integer N is congruent $\Leftrightarrow$ there is a point (x, y) on E with x, y rational and y non-zero.

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$$Ш(E/\mathbb{Q}) = Ker(H^1(\mathbb{Q}, E(\bar{\mathbb{Q}})) \to \prod_p H^1(\mathbb{Q}_p, E(\bar{\mathbb{Q}}_p))).$$

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• This group parametrizes some curves defined over $\mathbb{Q}$ that become isomorphic to the given elliptic curve E over extension fields of $\mathbb{Q}$

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- Numerically it is extremely difficult to calculate, but the BSD formula shows numerically that its order is remarkably small.

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We now turn to the next example in cyclotomic fields, due to Kummer.

Cyclotomic fields

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The *class group* of K is an arithmetic invariant attached to K, defined as the group of fractional ideals of the ring of integers in K modulo the principal ideals.

(b):Kummer's criterion

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• Kummer's criterion thus relates an arithmetic object namely the class group, to an analytic object, namely the special values of the Riemann zeta function.

The class group has a superficial analogy with the Tate-Shafarevich group.

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• Euler product expression
$L(E,s) = \prod_p (1 - a_p p^{-s} + (p^{1-2s}))^{-1}$, Re(s)> 3/2

• Dirichlet series expression $L(E,s) = \sum_{n=0}^{\infty} a_n/n^s$.

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• Dirichlet series expression $L(E,s) = \sum_{n=0}^{\infty} a_n/n^s$.

A deep and important result is that $L(E,s)$ has an analytic continuation to the whole complex plane.

Birch and Swinnerton-Dyer conjecture

Conjecture: The rank $g(E/\mathbb{Q})$ = order of vanishing of $L(E, s)$ at $s = 1$.
In particular, $E(\mathbb{Q})$ is infinite $\Leftrightarrow$ $L(E, s)$ vanishes at $s = 1$.

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- Extraordinary link between arithmetic objects and analytic objects.

For the CNP, with $N \equiv 5, 6, 7 \mod 8$, and for the curves $E : y^2 = x^3 - N^2 x,$ the theory of L-functions shows that $L(E, s)$ has an odd order zero at $s = 1$.

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From this perspective, the BSD conjecture seems very natural; can see how points of infinite order over $\mathbb{Q}$ give rise to a zero of multiplicity $g(E/\mathbb{Q})$, of a p-adic analogue of $L(E, s)$.

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- $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$; then G is a compact p-adic Lie group.

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$$\Lambda(G) = \lim_{\longleftarrow} \mathbb{Z}_p[G/G'].$$

Here G' runs over open normal subgroups of G and the inverse limit is taken with respect to the natural maps of the corresponding group rings.

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Fact: If G has no elements of order p (eg. $p > n+1$ in (iii)), then $\Lambda(G)$ is a Noetherian, Auslander regular domain with finite global dimension.

Examples

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(ii) $G \simeq \mathbb{Z}_p^d$, then $\Lambda(G) \simeq \mathbb{Z}_p[[T_1, \cdots, T_d]]$.
When G is commutative, study of modules over $\Lambda(G)$ is thus classical.

(iii) G open subgroup of $GL_n(\mathbb{Z}_p)$; $\Lambda(G)$ is more complicated; first investigated by Lazard.

Fact: If G has no elements of order p (eg. $p > n+1$ in (iii)), then $\Lambda(G)$ is a Noetherian, Auslander regular domain with finite global dimension.

This enables one to study modules over $\Lambda(G)$ quite generally using techniques from dimension theory and homological algebra.

Return to Arithmetic

Fundamental idea: Find a module over the Iwasawa algebra which simultaneously reflects both the arithmetic of E and the special values of the complex L-function attached to E.

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Classical descent theory tells us what arithmetic module we should consider over our p-adic Lie extension F_∞; namely the compact Pontryagin dual $X_p(E/F_\infty)$ of the p- primary Selmer group of E over F_∞.

This is a finitely generated module over the corresponding Iwasawa algebra $\Lambda(G)$, with $G = \mathrm{Gal}(F_\infty/F)$.

There is an exact sequence of $\Lambda(G)$-modules

$$0 \to \widehat{Ш\, E(F_\infty)}(p) \to X_p(E/F_\infty) \to \mathrm{Hom}(E(F_\infty) \otimes \mathbb{Z}_p, \mathbb{Z}_p) \to 0.$$

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This is exactly where the Main Conjecture intervenes.

Main Conjecture in Iwasawa theory

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• The statement of the Main Conjecture is that these two invariants are equal.

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$E(\mathbb{Q})$ infinite $\Rightarrow\ L(E,1) = 0.$
Assuming $Ш\,(E/\mathbb{Q})$ finite, $L(E,1) = 0 \Rightarrow E(\mathbb{Q})$ infinite.

Noncommutative case

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- In joint work with Coates and Schneider, we prove an analogue of the structure theorem for modules over certain noncommutative Iwasawa algebras.
- Later fromulated a precise Main conjecture for the non-commutative case in joint work with Coates, Fukaya, Kato and Venjakob.

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• Existence of an interesting Ore set makes such a localisation possible.

The noncommutative theory has a richer structure because of the existence of infinite families of self-dual irreducible Artin characters of G.

Applications and Examples

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Let m be any integer > 1, always assumed p-power free.

Define

$$L_n = \mathbb{Q}(m^{1/p^n}), \; K_n = \mathbb{Q}(\mu_{p^n}), \; F_n = \mathbb{Q}(\mu_{p^n}, m^{1/p^n}),$$

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Also known that for an elliptic curve E defined over $\mathbb{Q}$, the twisted complex L-functions $L(E, \rho_n, s)$ are entire; this uses deep results from Automorphic theory.

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Corresponding cusp form of weight 2 for $\Gamma_0(11)$ is

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This is very similar to other modular forms studied by Ramanujan who had different arithmetic questions in mind.

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Surprisingly, Iwasawa theory even gives sometimes an upper bound for the ranks.

Theorem: Assume m is any 7-power free integer with prime factors in the set $\{2, 3, 7\}$. Then for all $n = 1, 2, 3, \cdots$, we have

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This example is a special case of a more general theoretical result on $X_p(E/F_\infty)$; uses the philosophy of the noncommutative main conjecture.